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Masonry as Structured Continuum *

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Abstract. A structured continuum model is formulated to describe the behaviour of block masonry modelled as distinct rigid body systems with elastic interfaces. A correspondence between the two motions is obtained by postulating a relationship between the displacement fields of the continuum and the discrete models. The constitutive functions for the dynamic actions of the continuum are derived by equating the power of the two models.

Sommario. Viene presentato un modello di continuo con struttura atto a descrivere il comportamento meccanico di murature a blocchi, pensate come un sistema di corpi rigidi con contatti puntuali elastici. Il moto del sistema discreto e di quello continuo sono messi in relazione postulando una corrispondenza tra i campi di spostamento. Le funzioni costitutive delle azioni dinamiche del modello continuo sono ricavate uguagliando la potenza meccanica spesa in moti corrispondenti.

Key words: Masonry, Macroscopic characterization, Continuum mechanics.

1. Introduction

By using simple examples it can be shown that the description of the mechanical behaviour of block systems through a Cauchy continuum could be unsatisfactory in trivial cases (e.g. [1]). However, the observation that the block orientation plays a crucial role in the mechanical behaviour of the masonry requires a continuum of the Cosserat type, if the masonry has to be treated as a continuum. This idea is reinforced by the possibility of resorting to the Cosserat theory for the analysis of soil mechanics problems [2], where the 'blocks' are much smaller than in the masonry.

Despite such considerations, many investigators firmly believe the Cauchy continuum to be a good descriptor of the mechanical behaviour of the masonry; but some recent works [1], [3], [4] show that this is not the case.

The writers share the latter point of view; they point out that a more accurate continuum theory can really only be effective if a reliable way to describe the masonry as a continuum is found.

The purpose of this paper is to present a procedure to develop a Cosserat continuum which provides a description of the mechanical behaviour of the masonry with regular texture.

First, using a standard approach, the masonry is modelled as a discrete system of rigid blocks whose interactions are described by means of contact forces and couples. Second, a class of continua eligible to coarsely describe the behaviour of masonry is selected. Finally, by postulating the equivalence of the power expended for the two models, in correspondence

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with a class of regular motions, the constitutive functions for the continuum model are derived in terms of the geometrical and response properties of the rigid blocks model.

In order to highlight the methodological aspects, the procedure is developed in the framework of the linearized elasticity. However, it must be stressed that this procedure can be employed in more general contexts, even if the generalization is not trivial.

The analysis of a sample structure is carried out by using the discrete and the continuum models. The results obtained are in agreement with each other.

2. Continuum Model

This section outlines the main features of the theory of the structured continua. The interested reader is referred to [5], [6], and to [7] for an updated bibliography.

A substantial body $\mathcal{C}$ endowed with a local structure is a differential manifold whose shapes are in $\mathcal{E} \times SO(\mathcal{V})$ where $\mathcal{E}$ is a Euclidean point space, $\mathcal{V}$ is its translations space and $SO(\mathcal{V})$ is the rotation group of $\mathcal{V}$. A motion of $\mathcal{C}$ is defined as a function

$$\chi : \mathcal{C} \times \mathcal{I} \to \mathcal{E} \times SO(\mathcal{V}), \tag{1}$$

where $\mathcal{I} \subset \mathcal{R}$ is a time interval, $\mathcal{R}$ being the real line.

Any diffeomorphism $\kappa : \mathcal{C} \to \mathcal{E} \times SO(\mathcal{V})$ is called a reference placement, its image is called a reference shape, while $\mathcal{D} := \mathsf{p}(\kappa(\mathcal{C}))$, p being the natural projection into $\mathcal{E}$, is the restriction to $\mathcal{E}$ of the image of κ.

Given a function κ, the motion of $\mathcal{C}$ can be described in referential form by means of the functions $(\mathsf{x}, \mathbf{R})$ where $\mathsf{x} : \mathcal{D} \times \mathcal{I} \to \mathcal{E}$ and $\mathbf{R} : \mathcal{D} \times \mathcal{I} \to SO(\mathcal{V})$. Further $\mathbf{u}(X, t) := \mathsf{x}(X, t) - X$ denotes the displacement of the substantial point at X in $\mathcal{D}$. At a given time $t \in \mathcal{I}$, the fields $\mathsf{x}(\cdot, t)$, $\mathbf{R}(\cdot, t)$, describe the transplacement of $\mathcal{C}$ from the reference to the present shape, $\mathbf{R}$ accounting for the orientation of the local structure. The velocity at time t is defined by

$$\mathbf{v}, \mathbf{V} \qquad \mathbf{v} \in \mathcal{V},\ \mathbf{V} \in \text{Skew} \tag{2}$$

where $\mathbf{v} = \dot{\mathsf{x}} \circ \mathsf{x}^{-1}, \mathbf{V} = (\dot{\mathbf{R}} \circ \mathsf{x}^{-1})(\mathbf{R}^T \circ \mathsf{x}^{-1})$, the dot denotes the time differentiation and 'skew' the subspace of skew-symmetric tensors on $\mathcal{V}$. Now, we will introduce for a fixed time t, the fields

$$\begin{aligned} \mathbf{F} &:= \operatorname{grad} \mathsf{x} \\ \mathsf{F} &:= \operatorname{grad} \mathbf{R}, \end{aligned} \tag{3}$$

which we call the transplacement gradient. The polar decomposition theorem gives

$$\mathbf{F} = (\operatorname{orth} \mathbf{F})(\operatorname{psym} \mathbf{F}), \tag{4}$$

where 'orth' and 'psym' map $\mathbf{F}$ into its orthogonal and symmetric positive definite parts, respectively.

We introduce the following deformation measures

$$\begin{aligned} \mathbf{U} &:= \mathbf{R}^T(\operatorname{orth} \mathbf{F})(\operatorname{psym} \mathbf{F}) \\ \mathsf{U} &:= \mathbf{R}^T \circ \mathsf{F} \end{aligned} \tag{5}$$

and we state that the body undergoes a rigid transplacement if $\mathbf{U} = \mathbf{I}$ and $\mathsf{U} = 0$, that is if and only if $\operatorname{psym} \mathbf{F} = \mathbf{I}$, $\operatorname{orth} \mathbf{F} = \mathbf{R}$ and $\mathbf{R}$ are constant. The previous definition implies the assumption that the body undergoes a rigid transplacement if its projection on $\mathcal{E}$ is an isometry and the rotation of the microstructure corresponds to that of the body.

To describe the model from the dynamic point of view, we introduce the spatial fields of the contact force and couple $(\mathbf{t}, C)$ and the body force and couple $(\mathbf{b}, B)$, where $(\mathbf{t}, b \in \mathcal{V})$ and $(\mathbf{C}, \mathbf{B} \in \text{Skew})$. Introducing the stress tensor $\mathbf{T}$ and the couple-stress tensor C, a generalized Cauchy theorem can be stated for the contact force and couple as follows

$$\begin{aligned} \mathbf{t} &= \mathbf{Tn} \\ \mathbf{C} &= \mathsf{C}\mathbf{n}, \end{aligned} \tag{6}$$

$\mathbf{n}$ being the outward unit normal to $\partial\mathcal{F}$, where $\mathcal{F} := \mathsf{p}(\chi(\mathcal{C}, t))$. The balance equations are

$$\begin{aligned} \operatorname{div}\mathbf{T} + \mathbf{b} &= \mathbf{0} \\ \operatorname{div}\mathsf{C} - 2\operatorname{skew}\mathbf{T} + \mathbf{B} &= \mathbf{0}, \end{aligned} \tag{7}$$

while the mechanical power[1] is defined as follows

$$\mathcal{W} := \int_{\mathcal{F}} (\mathbf{b}\cdot\mathbf{v} + \tfrac{1}{2}\mathbf{B}\cdot\mathbf{V}) + \int_{\partial\mathcal{F}} (\mathbf{t}\cdot\mathbf{v} + \tfrac{1}{2}\mathbf{C}\cdot\mathbf{V}). \tag{8}$$

From (6) and (7), the following power formula is obtained

$$\mathcal{W} = \int_{\mathcal{F}} [\mathbf{T}\cdot(\operatorname{grad}\mathbf{v} - \mathbf{V}) + \tfrac{1}{2}\mathsf{C}\cdot\operatorname{grad}\mathbf{V}]. \tag{9}$$

We assume that $\mathcal{C}$ admits a natural state in which it stays at rest in a given shape forever, while all dynamic fields vanish. Assuming this shape as reference, we consider the motions of $\mathcal{C}$ that can be expressed in the form

$$\begin{aligned} \mathbf{u} &= \mathbf{w}\alpha + o(\alpha) \\ \mathbf{R} &= \mathbf{I} + \mathbf{W}\alpha + o(\alpha), \end{aligned} \tag{10}$$

where $\mathbf{W} \in \text{Skew}$, α is a real parameter and $\mathbf{u}, \mathbf{R}$ are analytic in α.

It should be noted that the assumption $\mathbf{R} = \mathbf{I}$ for $\alpha = 0$ means that in the given shape the orientation of the local structure is independent of the point at which it is attached. Making use of (10), from (5) one obtains

$$\begin{aligned} \mathbf{U} &= \mathbf{I} + \tilde{\mathbf{U}}\alpha + o(\alpha) \\ \mathsf{U} &= \tilde{\mathsf{U}}\alpha + o(\alpha), \end{aligned} \tag{11}$$

where $\tilde{\mathbf{U}}, \tilde{\mathsf{U}}$, the linearized deformation measures, result as

$$\begin{aligned} \tilde{\mathbf{U}} &= \mathbf{H} - \mathbf{W} \\ \tilde{\mathsf{U}} &= \mathsf{H}, \end{aligned} \tag{12}$$

where $\mathbf{H} = \operatorname{grad}\mathbf{w}$, $\mathsf{H} = \operatorname{grad}\mathbf{W}$ and the tilde denotes a differentiation with respect to α evaluated at $\alpha = 0$. Since we linearize near a state where $\mathbf{T} = 0$, $\mathsf{C} = 0$, the balance equations become

$$\begin{aligned} \operatorname{div}\mathbf{S} + \mathbf{b}_o &= \mathbf{0} \\ \operatorname{div}\mathsf{S} - 2\operatorname{Skew}\mathbf{S} + \mathbf{B}_o &= \mathbf{0}, \end{aligned} \tag{13}$$

[1] By replacing the velocity fields $(\mathbf{v}, \mathbf{V})$ with the corresponding tangent displacement fields, the mechanical power and the power formula assume the form of the external and internal work, respectively.

where $\mathbf{S} = \tilde{\mathbf{T}}$, $\mathsf{S} = \tilde{\mathbf{C}}$, $\mathbf{b}_o = \tilde{\mathbf{b}}$ and $\mathbf{B}_o = \tilde{\mathbf{B}}$. The linearized counterparts of the power and the power formula are

$$\begin{aligned} \mathcal{W}_o &= \int_{\mathcal{D}} (\mathbf{b}_o \cdot \dot{\mathbf{w}} + \tfrac{1}{2}\mathbf{B}_o \cdot \dot{\mathbf{W}}) + \int_{\partial\mathcal{D}} (\mathbf{t}_o \cdot \dot{\mathbf{w}} + \tfrac{1}{2}\mathbf{C}_o \cdot \dot{\mathbf{W}}) \\ &= \int_{\mathcal{D}} [\mathbf{S} \cdot (\dot{\mathbf{H}} - \dot{\mathbf{W}}) + \tfrac{1}{2}\mathsf{S} \cdot \dot{\mathsf{H}}] \end{aligned} \tag{14}$$

being $\mathbf{t}_o = \tilde{\mathbf{t}}$, $\mathbf{C}_o = \tilde{\mathbf{C}}$. Assuming that the body is made of elastic material, the linearized constitutive functions are

$$\begin{aligned} \mathbf{S} &= l_1(\tilde{\mathbf{U}}, \tilde{\mathsf{U}}) \\ \mathsf{S} &= l_2(\tilde{\mathbf{U}}, \tilde{\mathsf{U}}), \end{aligned} \tag{15}$$

where l_1, l_2 are linear functions and $l_1(\mathbf{0}, 0) = \mathbf{0}, l_2(\mathbf{0}, 0) = \mathsf{0}$.

3. Discrete Model

In this section we present a discrete model of block masonry with regular netting in the framework of the linearized elasticity.

Let us assume that each block is a rigid body. Given a reference shape for the masonry, the motion of one of its blocks, $\mathcal{A}$, will be described by means of the functions $\mathbf{u}^a, \mathbf{R}^a$; the former denoting the displacement of the block centre g^a, the latter the block orientation.

As already done for the continuum, we will assume that $\mathbf{u}^a, \mathbf{R}^a$, are analytic functions of a parameter α, so that we put

$$\begin{aligned} \mathbf{u}^a &= \mathbf{w}^a \alpha + o(\alpha) \\ \mathbf{R}^a &= \mathbf{I} + \mathbf{W}^a \alpha + o(\alpha), \end{aligned} \tag{16}$$

where $\mathbf{W}^a$ is a skew tensor, $\mathbf{w}^a = \mathbf{w}^a(g^a)$, and the time arguments have been omitted.

Since the block is a rigid body, the linear part of its motion will be described in the form

$$\mathbf{w}^a(\cdot) = \mathbf{w}^a + \mathbf{W}^a((\cdot) - g^a). \tag{17}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two blocks and $(P^a \in \mathcal{A}, P^b \in \mathcal{B})$ a pair of substantial points—that we call a 'test pair'—whose positions in a given reference shape are p^a, p^b respectively. As 'test strain measures' for the pair (p^a, p^b) we assume the following quantities

$$\begin{aligned} \mathbf{w}_p &:= \mathbf{w}^b(p^b) - \mathbf{w}^a(p^a) \\ \mathbf{W}_p &:= \mathbf{W}^b - \mathbf{W}^a \end{aligned} \tag{18}$$

that, by means of (17), can be rewritten as follows

$$\mathbf{w}_p = \mathbf{w}^b - \mathbf{w}^a + \mathbf{W}^b(p - g^b) - \mathbf{W}^a(p - g^a). \tag{19}$$

Furthermore, we assume that a test pair is introduced only for couples of adjacent blocks interacting with each other by means of contact actions. The interaction is described by a force and a couple. In particular, given two blocks $\mathcal{A}$ and $\mathcal{B}$, the force and the couple exerted by $\mathcal{B}$ on $\mathcal{A}$, with regard to the test pair (p^a, p^b), are expressed by the following constitutive functions

$$\begin{aligned} \mathbf{t}_p &= \mathbf{K}_p \mathbf{w}_p \\ \mathbf{C}_p &= \mathsf{K}_p \mathbf{W}_p. \end{aligned} \tag{20}$$

Let us consider now a part $\mathcal{P}$ of the wall, made of n blocks. The mechanical power of the forces acting on $\mathcal{P}$ can be expressed in the form

$$\pi = \sum_i (\mathbf{f}^i \cdot \dot{\mathbf{w}}^i + \tfrac{1}{2}\mathbf{M}^i \cdot \dot{\mathbf{W}}^i), \tag{21}$$

where $\mathbf{f}^i$ is the linear part of the contact and body forces on $\mathcal{P}$ relative to the block i, $\mathbf{M}^i$ the moment of $\mathbf{f}^i$ with respect to g^i, and i ranges from 1 to n.

From (21) the following power formula can easily be derived

$$\pi = \sum_p \pi_p, \qquad \pi_p = \mathbf{t}_p \cdot \dot{\mathbf{w}}_p + \tfrac{1}{2}\mathbf{C}_p \cdot \dot{\mathbf{W}}_p, \tag{22}$$

the range of p being the number of the test pairs in $\mathcal{P}$.

4. Identification of the Continuum Model

The purpose of this section is to show that the continuum model presented in Section 2 can be seen as a coarse descriptor of the discrete model introduced in Section 3. A procedure based on the assumption of a correspondence between the motions of the continuum and the discrete models is therefore proposed, following an approach similar to the one adopted in [8].

Given a shape for $\mathcal{C}$ and a place $x \in \mathcal{D}$, let $\mathcal{U}$ denote an open neighbourhood of x. Let us assume that the linear part of a generic motion (see (10)), restricted to $\mathcal{U}$, can be sufficiently approximated by the functions

$$\begin{aligned} \mathbf{w}(q,t) &= \mathbf{w}(x,t) + \mathbf{H}(x,t)(q-x) \\ \mathbf{W}(q,t) &= \mathbf{W}(x,t) + \mathsf{H}(x,t)(q-x) \end{aligned} \tag{23}$$

for any $q \in \mathcal{U}$.

If we assume that the wall has a modular structure, the motion of the module can be related to the motion of $\mathcal{U}$ by postulating that

$$\begin{aligned} \mathbf{w}^a &= \mathbf{w}(x) + \mathbf{H}(x)(g^a - x) \\ \mathbf{W}^a &= \mathbf{W}(x) + \mathsf{H}(x)(g^a - x). \end{aligned} \tag{24}$$

Note that time arguments have been omitted in (24) and will be omitted in the following expressions. Finally, by using (24) and (17), expressions (18) can be rewritten as follows

$$\begin{aligned} \mathbf{w}_p &= \mathbf{H}(x)(g^b - g^a) - \mathbf{W}(x)(g^b - g^a) \\ &\quad + [\mathsf{H}(x)(g^b - x)](p^b - g^b) - [\mathsf{H}(x)(g^a - x)](p^a - g^a) \\ \mathbf{W}_p &= \mathsf{H}(x)(g^b - g^a). \end{aligned} \tag{25}$$

4.1. Constitutive Functions for the Contact Actions

Using the motion correspondence proposed above, it is possible to obtain an expression of the power of the contact actions in the discrete model in terms of the kinematical quantities pertaining to those of the continuum. With regard to a generic test pair (p^a, p^b), from (22) and (25) one obtains

$$\begin{aligned}\pi_p &= \mathbf{t}_p \cdot \Big\{\dot{\mathbf{H}}(x)(g^b - g^a) - \dot{\mathbf{W}}(x)(g^b - g^a)\\ &\quad +[\dot{\mathsf{H}}(x)(g^b - x)](p^b - g^b) - [\dot{\mathsf{H}}(x)(g^a - x)](p^a - g^a)\Big\}\\ &\quad +\tfrac{1}{2}\mathbf{C}_p \cdot \dot{\mathsf{H}}(x)(g^b - g^a). \end{aligned} \tag{26}$$

By performing simple algebra, the above expression can be rewritten in the form

$$\begin{aligned}\pi_p &= [\dot{\mathbf{H}}(x) - \dot{\mathbf{W}}(x)] \cdot [\mathbf{t}_p \otimes (g^b - g^a)]\\ &\quad +\tfrac{1}{2}\dot{\mathsf{H}}(x) \cdot \Big\{2\mathbf{t}_p \otimes [(p^b - g^b) \otimes (g^b - x) - (p^a - g^a) \otimes (g^a - x)]\\ &\quad +\mathbf{C}_p \otimes (g^b - g^a)\Big\}. \end{aligned} \tag{27}$$

The constitutive functions for the contact actions can thus be obtained by requiring that the stress power density at x for the continuum model, given by $(14)_2$, is equal to

$$\sum_p \frac{\pi_p}{V}, \forall\, (\dot{\mathbf{H}} - \dot{\mathbf{W}}),\ \dot{\mathsf{H}}, \tag{28}$$

where V denotes the volume of the module, the summation is extended to all the test pairs appearing in the selected module thought as being 'centred' in x, while π_p has been expressed in terms of $\mathbf{W}, \mathbf{H}, \mathsf{H}$ via (25). From the above expression, one obtains

$$\begin{aligned}\mathbf{S} &= \frac{1}{V}\sum_p \mathbf{t}_p \otimes (g^b - g^a),\\ \mathsf{S} &= \frac{1}{V}\sum_p \Big\{2\mathbf{t}_p \otimes [(p^b - g^b) \otimes (g^b - x) - (p^a - g^a) \otimes (g^a - x)]\\ &\quad +\mathbf{C}_p \otimes (g^b - g^a)\Big\}, \end{aligned} \tag{29}$$

where

$$\begin{aligned}\mathbf{t}_p &= \mathbf{K}_p \Big\{[\mathbf{H}(x) - \mathbf{W}(x)](g^b - g^a)\\ &\quad +[\mathsf{H}(x)(g^b - x)](p^b - g^b) - [\mathsf{H}(x)(g^a - x)](p^a - g^a)\Big\},\\ \mathbf{C}_p &= \mathsf{K}_p\, [\mathsf{H}(x)(g^b - g^a)]. \end{aligned} \tag{30}$$

4.2. Constitutive functions for the body actions

Following the same procedure adopted for contact actions, the constitutive functions for body actions can be constructed. The cases of gravity and inertia actions are considered below.

4.2.1. *Gravity actions*

Let $\mathbf{f}^a$ be the weight of a single block $\mathcal{A}$ of the module. The power expended by the body forces acting on the module, in view of (24), can be put in the form

$$\begin{aligned}\pi &= \sum_a \mathbf{f}^a \cdot \dot{\mathbf{w}}^a\\ &= \dot{\mathbf{w}}(x) \cdot \sum_a \mathbf{f}^a + \dot{\mathbf{H}}(x) \cdot \sum_a \mathbf{f}^a \otimes (g^a - x). \end{aligned} \tag{31}$$

If x coincides with the centre of gravity of the module, $\sum_a \mathbf{f}^a \otimes (g^a - x) = 0$ and (31) is

$$\pi = \dot{\mathbf{w}}(x) \cdot \sum_a \mathbf{f}^a. \tag{32}$$

The density of the power expended by body actions on the continuum model, for any velocity field characterized by $\dot{\mathbf{w}}$ and $\dot{\mathbf{W}}$, is given by the first integrand in $(14)_1$. Assuming that such density is equal to π/V, the body actions of the continuum model, as a function of the gravity loads acting on each block of the module, result as

$$\begin{aligned} \mathbf{b}_o &= \frac{1}{V}\sum_a \mathbf{f}^a = \rho\mathbf{g} \qquad \rho = \frac{1}{V}\sum_a M^a \\ \mathbf{B}_o &= \mathbf{0}, \end{aligned} \tag{33}$$

where M^a is the mass of the block $\mathcal{A}$, and $\mathbf{g}$ is the gravity acceleration vector.

4.2.2. *Inertial actions*

The power expended by the inertial actions on the discrete model for any velocity field characterized by $\dot{\mathbf{w}}^a$, $\dot{\mathbf{W}}^a$ is

$$\pi = -\frac{1}{V}\int_{\mathcal{D}} \sum_a [M^a\, \ddot{\mathbf{w}}^a \cdot \dot{\mathbf{w}}^a + \tfrac{1}{2}\mathbf{M}^a \cdot \dot{\mathbf{W}}^a], \tag{34}$$

where

$$\mathbf{M}^a = \mathbf{E}^a(\ddot{\mathbf{W}}^a + \dot{\mathbf{W}}^{a^2})^T - (\ddot{\mathbf{W}}^a + \dot{\mathbf{W}}^{a^2})\mathbf{E}^a \tag{35}$$

is the time derivative of the angular momentum and $\mathbf{E}^a$ is the Euler tensor for the block $\mathcal{A}$.

By identifying the motions of the discrete and the continuum models by means of (24), we can write the power as follows

$$\begin{aligned} \pi = & -\dot{\mathbf{w}}(x) \cdot \frac{1}{V}\int_{\mathcal{D}} \sum_a M^a\, \ddot{\mathbf{w}}^a - \dot{\mathbf{H}}(x) \cdot \frac{1}{V}\int_{\mathcal{D}} \sum_a M^a\, \ddot{\mathbf{w}}^a \otimes (g^a - x) \\ & -\tfrac{1}{2}\dot{\mathbf{W}}(x) \cdot \frac{1}{V}\int_{\mathcal{D}} \sum_a \mathbf{M}^a - \tfrac{1}{2}\dot{\mathsf{H}}(x) \cdot \frac{1}{V}\int_{\mathcal{D}} \sum_a \mathbf{M}^a \otimes (g^a - x) \end{aligned} \tag{36}$$

and, as a consequence of the divergence theorem

$$\begin{aligned} \pi = & -\dot{\mathbf{w}}(x) \cdot \frac{1}{V}\int_{\mathcal{D}} \sum_a M^a\{\ddot{\mathbf{w}}^a - \operatorname{div}[\ddot{\mathbf{w}}^a \otimes (g^a - x)]\} \\ & -\tfrac{1}{2}\dot{\mathbf{W}}(x) \cdot \frac{1}{V}\int_{\mathcal{D}} \sum_a \{\mathbf{M}^a - \operatorname{div}[\mathbf{M}^a \otimes (g^a - x)]\} \\ & -\dot{\mathbf{w}}(x) \cdot \frac{1}{V}\int_{\partial\mathcal{D}} \sum_a M^a[\ddot{\mathbf{w}}^a \otimes (g^a - x)]\mathbf{n} \\ & -\tfrac{1}{2}\dot{\mathbf{W}}(x) \cdot \frac{1}{V}\int_{\partial\mathcal{D}} \sum_a [\mathbf{M}^a \otimes (g^a - x)]\mathbf{n}, \end{aligned} \tag{37}$$

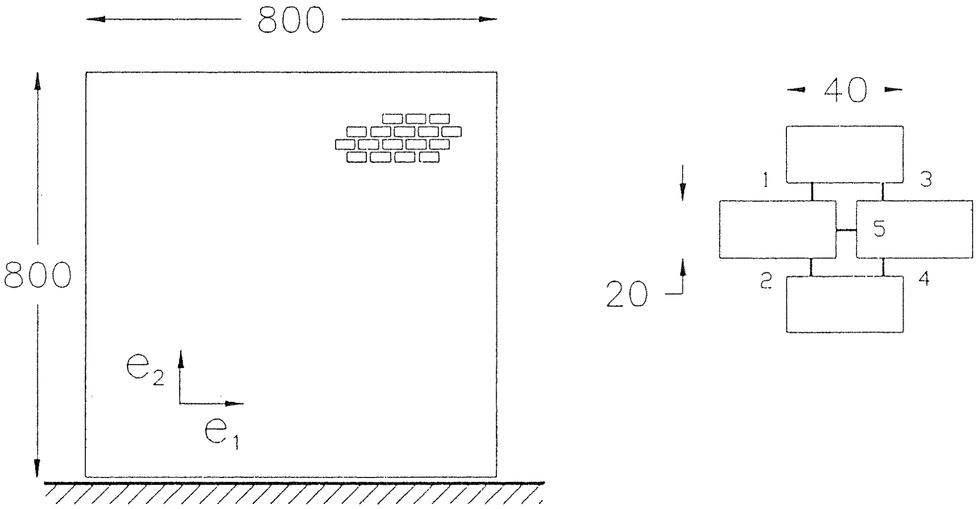

Fig. 1. Sketch of the test problem.

where $\mathbf{n}$ is the outward unit normal field on $\partial\mathcal{D}$. Finally, as already done, we postulate that the power density of the continuum, at $x \in \mathcal{D}$, equals π/V, and we obtain the following expressions for the densities of the inertial body actions in $\mathcal{D}$

$$\begin{aligned} \mathbf{b}_o &= -\frac{1}{V}\sum_a M^a\{\ddot{\mathbf{w}}^a - \mathrm{div}[\ddot{\mathbf{w}}^a \otimes (g^a - x)]\} \\ \mathbf{B}_o &= -\frac{1}{V}\sum_a \{\mathbf{M}^a - \mathrm{div}[\mathbf{M}^a \otimes (g^a - x)]\} \end{aligned} \tag{38}$$

together with the inertial surface actions on $\partial\mathcal{D}$

$$\begin{aligned} \mathbf{S} &= -\frac{1}{V}\sum_a M^a[\ddot{\mathbf{w}}^a \otimes (g^a - x)] \\ \mathsf{S} &= -\frac{1}{V}\sum_a [\mathbf{M}^a \otimes (g^a - x)]. \end{aligned} \tag{39}$$

Finally, by performing time derivatives of (24), (38) and (39) can be rewritten in terms of continuum fields.

5. Analysis of a Sample Problem

In this section a square masonry wall resting on a fixed rigid body is considered. Its reference shape, the module and the base adopted are shown in Figure 1. The wall is acted upon by a constant field of body forces whose density has components $f_1 = 5\ 10^{-3}$, $f_2 = -10^{-2}$.

As a first step, the model introduced in Section 3 is exploited to describe the behaviour of the wall. The blocks lying on the fixed body are considered to be embedded in it. For each test pair the constitutive functions (20) are specified as follows

$$(\mathbf{K}_p) = \begin{pmatrix} 5\ 10^4 & 0 \\ 0 & 10^4 \end{pmatrix}, \quad K_p = 5\ 10^6, \tag{40}$$

K_p being the sole independent element of K_p.

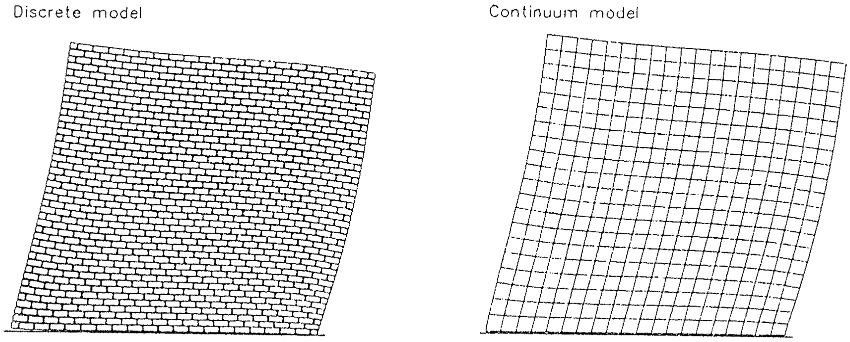

Fig. 2. Deformed shapes of the wall.

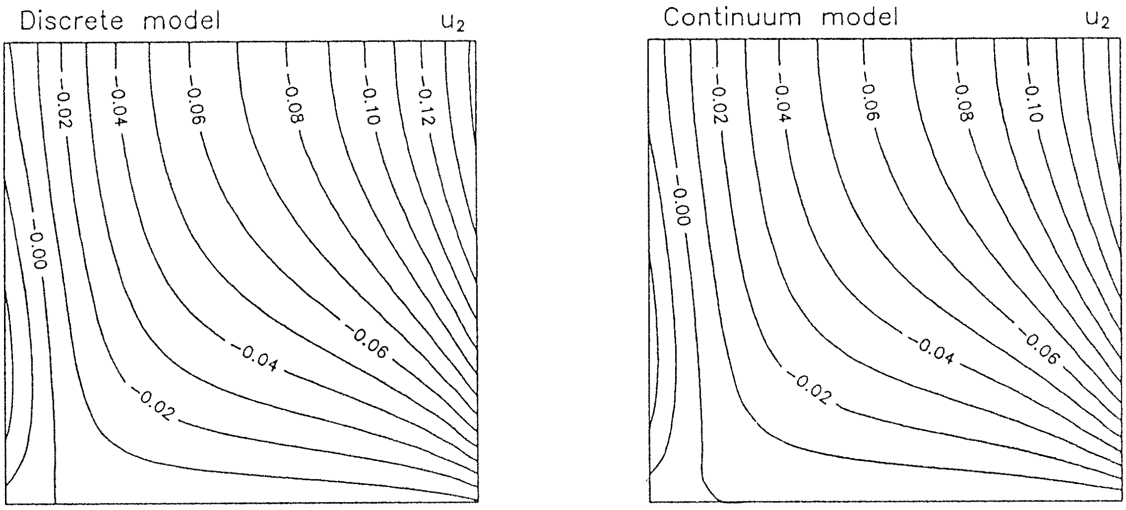

Fig. 3. Contour lines of the component along $\mathbf{e}_2$ of the displacement fields.

Secondly, by performing the procedure proposed in Section 4, the continuum model is constructed and the constitutive functions for the contact actions result: $S_{11} = 3\ 10^4 U_{11}, S_{22} = 2.5\ 10^4\ U_{22}, S_{12} = 5\ 10^3\ U_{12}, S_{21} = 3\ 10^4\ U_{21}, \mathsf{S}_{121} = 5\ 10^6\ \mathsf{U}_{121}, \mathsf{S}_{122} = 2.5\ 10^6\ \mathsf{U}_{122}$ (the components of a tensor are defined by: $A_{ij} = (\mathbf{A}\mathbf{e}_j) \cdot \mathbf{e}_i$ and $\mathsf{A}_{ijk} = [(\mathsf{A}\mathbf{e}_k)\mathbf{e}_j] \cdot \mathbf{e}_i$).

It is worth noting that the above constitutive functions result in being strictly diagonal as a consequence of the chosen module and of the simple form (40) assumed for (20). Less trivial assumptions for the blocks interactions can lead to more refined constitutive functions for the continuum. For instance, by putting $K_{p12} = K_{p21} = a \neq 0$ for the test pairs (1) and (2) (see Figure 1) and $K_{p12} = K_{p21} = -a$ for the test pairs (3) and (4), one obtains a material characterized by non-diagonal constitutive functions. It can easily be seen that such a material exhibits the Poisson effect, while the stress $\mathbf{S}$ and the couple stress S depend both on the microrotation gradient H and on the strain $\mathbf{H} - \mathbf{W}$.

Finally, both the discrete and the continuum problems have been solved; the latter by means of the F.E. technique using triangular elements. Each triangle has nine degrees of freedom: six corner translations and three corner rotations. For the nodes lying along the constrained edge of the wall both the displacements and the rotations have been restrained.

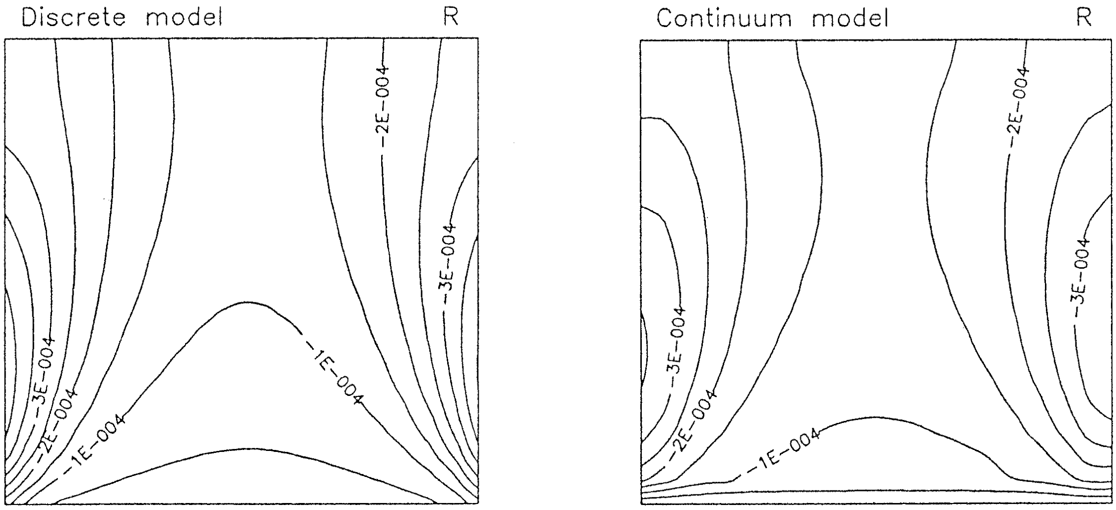

Fig. 4. Contour lines of the microrotation fields.

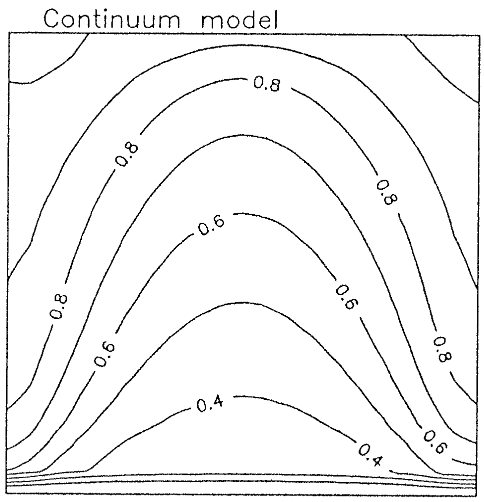

Fig. 5. Contour lines of the ratio between the microrotation and the macrorotation fields.

Figures 2 and 3 show some results obtained in the two cases. The shape of the wall under loading is shown in Figure 2 (the line of embedded blocks does not appear in the picture). Figure 3 shows the contour lines for the component along $\mathbf{e}_2$ of the displacement field.

Since the rotation of the continuum microstructure was assumed as the counterpart of the block rotation in the discrete model, the value of both the quantities for the case in hand is illustrated in Figure 4 in the form of a contour plot.

Finally, Figure 5 shows the contour lines for the ratio between the rotation of the microstructure and the rotation of the macrostructure (the skew part of the displacement gradient). This ratio can be assumed as a measure for the relevance of the microstructure.

The relevance of the microstructure method, can be confirmed following another way. One can find a Cauchy type solution for the problem at hand. Such a solution can then be compared with the one found in this paper. Such analysis is out of the scope of the present work and is the object of a forthcoming paper [9].

6. Conclusions

A procedure relevant for identifying a Cosserat continuum for block masonry has been presented within the framework of linearized elasticity. Explicit relations of the constitutive functions of the contact, gravity and inertia actions for the continuum, have been obtained. A two-dimensional sample problem was analysed showing the response of the continuum model in agreement with that of the continuum.

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