J. Phys. IV France 8 (1998)

A continuum model with microstructure for materials with flaws and inclusions

P. Trovalusci and G. Augusti

Università di Roma "La Sapienza", Dipartimento di Ingegneria Strutturale e Geotecnica, via Eudossiana 18, 00184 Roma, Italy

Abstract: A continuum endowed with affine microstructure is adopted for the macroscopic description of fiber composite materials. The microstructure is made of a rigid and of a deformable local structure. The former represents the fibers of the composite, perceived as rigid inclusions. The latter accounts for the presence of distributed flaws, considered as slit microcracks. In the framework of a degree one theory, a formula for the mechanical power is derived from a discrete microscopic model using an integral procedure of equivalence. Constitutive elastic stress-strain relationships, accounting for the geometry of the internal phases, are identified. The balance equations for both the continuum macro and micro-actions are derived from the axiom of vanishing power and of invariance of power under change of observer. It is also shown that the material symmetries are preserved in the transition from fine to gross description.

1. INTRODUCTION

The mechanical behaviour of composite materials made of short, stiff and strong fibers embedded in a more deformable matrix is strongly influenced by the size, the orientation and the disposition of the fibers as well as by the mechanical properties of the constituents. Moreover, microflaws due to manufacturing defects or lack of cohesion can be present in the matrix. We therefore propose a macroscopic mechanical model of such composites which considers the presence of two internal material phases, the inclusions and the microcracks, with their respective geometry.

While for composite materials a gross modelling is generally necessary, the effectiveness of this description is conditioned by the choice of the specific continuum model. The aim of this paper is to indicate a way to select the macroscopic model. Over the last decades the possibility to give a macroscopic description for heterogeneous elastic media, accounting for the geometry of the microstructure, has been largely investigated within the framework of the homogeneization theory; either employing averaging methods, including statistical methods,[1] or, in order to model complex spatial interaction effects, resorting to techniques of asymptotic expansions up to the higher orders which involve the successive gradients of deformations (e.g. [2]). The relations among the gradient theories, the ensemble averaging theories and the so-called non local theories have been known for some times (e.g. [3]). These theories represent an effective way to by-pass the inadequacies of simple material models however, difficulties arise when a thermomechanical process is considered. For example, Gurtin [4] demonstrated that the constitutive assumption of a Cauchy material of *grade N* is incompatible with the second law of thermodynamics. More recently, Dunn and Serrin [5] showed that, to preserve the Clausius-Duhem inequality, an additional quantity must be introduced in the energy balance. From balance considerations, Capriz [6] proved that this approach is analogous to introducing a continuum with microstructure.

We think that to adopt a continuum with structure is a convenient way to retain memory of the fine organization of the material at the mesoscopic level, accounting for the higher order effects and avoiding the above mentioned difficulties. In this context the problem is the selection of an appropriate microstructure, that is the recognition of the additional kinematical and dynamical descriptors to be introduced, and the identification of the constitutive parameters.

As is not unusual in continuum mechanics, consideration based on kinematics of lattice systems have been often employed to derive the field equations for materials endowed with structure [8, 9, 10]. In this paper, following Ericksen [10], we use lattice kinematics to construct a macroscopic model that develops the same mechanical power as the microscopic model in any 'corresponding' velocity fields. Once a discrete model as detailed as necessary is defined, the natural request of preservation of the mechanical power provides a tool to select the most appropriate continuum model. This is the only model for which explicit constitutive functions for all the continuum stress measures can be identified, without constitutive prescriptions which alter the adopted fine model.[1]

The gross model thus derived is a continuum with *affine* microstructure [7]; it can include micromotions and is described by three independent homogeneous fields: the macroscopic displacement field and two microscopic kinematical fields. The two microscopic fields correspond respectively to a rigid and a deformable local structure. The former is a tensor field representing the rotations of the individual fibers, the latter is a vector field representing the difference between the actual displacement field of a body with distributed microflaws and the displacement field of the body without flaws. The convenience of introducing a rigid local structure (Cosserat) to describe materials in which the size and the mutual rotations of the particles with dimensions are not negligible has been shown several times [12, 13, 14]. The deformable microstructure allows to describe the occurrence of damage as an increase of the actual state of strain and stress instead of a decrease of the overall stiffness, which would in turn require internal variables (e.g. [15]). These two separate local structures have been already proposed to model masonry-like materials [16, 17] and microcracked materials [18]. In this paper the formulation is extended to multi-phase materials and their coupling is investigated. Moreover, in order to describe the appropriate geometry of the original fine model, the correspondence between the material symmetries of the discrete and the continuum model is examined.

2. THE LATTICE SYSTEM

We assume that the body is characterized by a continuous distribution of fibers, much stiffer than the matrix in which they are embedded, and of slit microcracks which are considered open, stationary and with blunt edges. The discrete model adopted for the fine description of such a material is made of two superposed lattice systems. The former consists of rigid particles of assigned shape, representing the fibers, connected in pair by linear elastic links, representing the matrix. The latter is made of slits, of arbitrary shape and a predominant dimension, connected one to another by non linear elastic links which can carry only axial forces. The two lattices are linked by rigid bonds. The second lattice is only an expedient for transmitting to the matrix the additional forces due to the presence of microcracks: in this sense the slits, whose stiffness depends on the surrounding elastic field, represent the microcraks.

Here below we limit the analysis to a linearized static theory so that the velocity fields stand for the displacement fields.

Let $\mathcal{A}$ and $\mathcal{B}$ be two particles centred respectively at a and b, and $\mathcal{H}$ and $\mathcal{K}$ be two slits with centres h and k respectively. The vectors $\mathbf{w}^a$ and $\mathbf{w}^b$ denote the velocity of a and b respectively, while the skew-symmetric second order tensors $\mathbf{W}^a$ and $\mathbf{W}^b$ denote the angular velocities of the two particles. For each pair i of adjacent particles we define the strain measures

$$\mathbf{w}_i = \mathbf{w}^b - \mathbf{w}^a + \mathbf{W}^a(p_i^a - a) - \mathbf{W}^b(p_i^b - b)\,, \qquad \mathbf{W}_i = \mathbf{W}^a - \mathbf{W}^b\,, \tag{1}$$

where p_i^a and p_i^b are two points, on $\mathcal{A}$ and $\mathcal{B}$ respectively, through which the particles interact.

The stress on a crack within an elastic body depends on the strain of its boundary and on the deformation of the neighbouring defects. We assume that the slits can deform only in

[1] In a forthcoming paper [11], it is shown that these prescriptions are either internal constraints or constraints on the geometry of the internal phases.

a direction normal to the direction, $\mathbf{n}^h$, of their major axis: the mean displacement jump, $\mathbf{d}^h$, between two opposite surfaces of a slit $\mathcal{H}$ and the difference between two mean displacements jumps, $\mathbf{d}_j = \mathbf{d}^h - \mathbf{d}^k$, of two interacting slits $\mathcal{H}$ and $\mathcal{K}$, are defined as further strain measures of the lattice system.

The external forces and couples that act on the material particles are represented respectively by the vector $\mathbf{f}^a$ and the skew-symmetric second order tensor $\mathbf{M}^a$. The forces and the couples that $\mathcal{B}$ exerts on $\mathcal{A}$ are represented by the vector $\mathbf{t}_i^a$ and the skew-symmetric second order tensor $\mathbf{C}_i^a$. The force on $\mathcal{H}$ due to the crack opening displacement, $\mathbf{d}^h$, is represented by the vector $\mathbf{z}_o^h$. Due to this displacement, each slit can interact with the adjacent particles and with the neighbouring defects. The vector $\mathbf{r}_l^a$ represents the action transmitted by a slit $\mathcal{H}$ to the particle $\mathcal{A}$, while the vector $\mathbf{z}_j^h$ is the action that the slit $\mathcal{K}$ exerts on $\mathcal{H}$.

The balance equations of actions for each particle $\mathcal{A}$ are

$$\sum_{i=1}^{N^a}\mathbf{t}_i^a + \sum_{l=1}^{L^a}\mathbf{r}_l^a + \mathbf{f}^a = \mathbf{0}\,, \qquad \sum_{i=1}^{N^a}[\mathbf{C}^a + (p_i^a - a)\otimes\mathbf{t}_i^a - \mathbf{t}_i^a\otimes(p_i^a - a)] + \mathbf{M}^a = \mathbf{0}\,, \tag{2}$$

where N^a and L^a are respectively the number of the particles and of the slits interacting with the particle $\mathcal{A}$.[2] For each link i between two particles the balance equations are

$$\mathbf{t}_i^a + \mathbf{t}_i^b = \mathbf{0}\,, \qquad \mathbf{C}_i^a + \mathbf{C}_i^b + (p_i^a - p_i^b)\otimes\mathbf{t}_i^a - \mathbf{t}_i^a\otimes(p_i^a - p_i^b) = \mathbf{0}\,. \tag{3}$$

The balance equations of forces over each slit $\mathcal{H}$ is

$$\sum_{j=1}^{L^h}\mathbf{z}_j^h + \mathbf{z}_o^h + \sum_{l=1}^{N^h}\mathbf{r}_l^h = \mathbf{0}\,, \tag{4}$$

where L^h and N^h are respectively the number of the slits and of the particles interacting with the slit $\mathcal{H}$. Moreover,

$$\mathbf{z}_j^h + \mathbf{z}_j^k = \mathbf{0}\,, \qquad \mathbf{r}_l^a + \mathbf{r}_l^h = \mathbf{0}\,, \tag{5}$$

for each pair of slits j and of rigid bonds l respectively. Finally, we have the following constitutive prescriptions on the internal lattice actions

$$(\mathbf{I} - \mathbf{n}_j\otimes\mathbf{n}_j)\,\mathbf{z}_j = \mathbf{0}\,, \qquad (\mathbf{n}^h\otimes\mathbf{n}^h)\,\mathbf{z}_o^h = \mathbf{0}\,, \tag{6}$$

where $\mathbf{I}$ is the second order identity tensor and $\mathbf{n}_j = (h-k)/\parallel h-k \parallel$.

If the material can be considered statistically homogeneous,[3] a unit cell M, denoted module, can be individuated. From balance equations (2-5) the mean power over the volume $V(\mathsf{M})$ of the module can be written in terms of its internal actions

$$\pi(\mathbf{w}_i, \mathbf{W}_i, \mathbf{d}^h, \mathbf{d}_j) = \frac{1}{V(\mathsf{M})}\{\sum_{i=1}^{N}[\mathbf{t}_i\cdot(\mathbf{w}_i - \mathbf{W}^b(p_i^a - p_i^b)) + \frac{1}{2}\mathbf{C}_i^a\cdot\mathbf{W}_i] + \sum_{h=1}^{L}\mathbf{z}_o^h\cdot\mathbf{d}^h + \sum_{j=1}^{M}\mathbf{z}_j\cdot\mathbf{w}_j\}\,, \tag{7}$$

where N, L and M are respectively the number of the pairs of interacting particles, the number of the slits and the number of the pair of interacting slits of the module, while $\mathbf{t}_i^a = -\mathbf{t}_i^b = \mathbf{t}_i$ and $\mathbf{z}_j^h = -\mathbf{z}_j^k = \mathbf{z}_j$.

As first step, let us assume linear elastic response functions for the interactions between particles and for the forces due to the crack opening displacements, while, in accord with the Barenblatt's theory ([19], cap. IV), the constitutive functions for the interactions between slits are non linear elastic:

$$\mathbf{t}_i = \mathbb{K}_i^t\mathbf{w}_i\,, \qquad \mathbf{C}_i = \mathbb{K}_i^c\mathbf{W}_i\,, \qquad \mathbf{z}_o^h = \mathbb{D}^h\mathbf{d}^h\,, \qquad \mathbf{z}_j = \mathcal{D}\parallel\mathbf{d}^h\parallel\parallel\mathbf{d}^k\parallel\frac{(h-k)}{\parallel h-k\parallel}\,. \tag{8}$$

[2]Let a and b be two vector, the axial vector of the skew-symmetric tensor $\mathbf{a}\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{a}$ is $\mathbf{a}\times\mathbf{b}$.
[3]See [2] for a discussion about homogenizable situations.

The components of the second order tensors $\mathbb{K}$ and $\mathbb{D}$ and the fourth order tensor $\mathbb{K}$, and the scalar $\mathcal{D}$, defined in [17] and [18], depend on the elastic constants of the matrix and on the geometry of the microstructure.

3. IDENTIFICATION OF THE EQUIVALENT CONTINUUM

In order to identify the equivalent continuum model, hypotheses of regularity of the kinematical descriptors must be introduced [16]. Let us identify the region of the Euclidean space $\mathcal{E}$ occupied by the module with the neighbourhood $\mathcal{N}$ of a position x. Consistently with the molecular theory [10], accounting for local kinematical discrepancies and for short range interactions, it suffices to consider homogeneous deformations in $\mathcal{N}$. Then we select the admissible velocities of the module assuming the affine representations

$$\mathbf{w}^a = \mathbf{w}(x) + \mathbf{H}(x)(a-x)\,, \qquad \mathbf{W}^a = \mathbf{W}(x) + \mathsf{H}(x)(a-x)\,, \qquad \mathbf{d}^h = \mathbf{d}(x) + \mathbf{N}(x)(a-x)\,, \quad (9)$$

where $\mathbf{w}(x)$, $\mathbf{W}(x)$ and $\mathbf{d}(x)$ are smooth fields and $\mathbf{H} = \operatorname{grad}\mathbf{w}$, $\mathsf{H} = \operatorname{grad}\mathbf{W}$, $\mathbf{N} = \operatorname{grad}\mathbf{d}$. Therefore the strain measures of the lattice can be expressed in terms of $\mathbf{H}-\mathbf{W}$, H, $\mathbf{d}$ and $\mathbf{N}$, while the mean power of M is

$$\begin{aligned}
\pi(\mathbf{w},\mathbf{W},\mathbf{d}) &= \frac{1}{V(\mathsf{M})}\{(\mathsf{H}(x)-\mathbf{W}(x))\cdot\sum_{i=1}^{N}\mathbf{t}_i\otimes(a-b) \\
&+ \frac{1}{2}\mathsf{H}(x)\cdot\{\sum_{i=1}^{N}\{2\mathbf{t}_i\otimes[(p^a-a)\otimes(a-x)-(p^b-b)\otimes(b-x)]+\mathbf{C}_i^a\otimes(a-b)\} \\
&+ \mathbf{d}\cdot\sum_{h=1}^{L}\mathbf{z}_o^h+\mathbf{N}\cdot[\sum_{h=1}^{L}\mathbf{z}_o^h\otimes(h-x)+\sum_{j=1}^{M}\mathbf{z}_j\otimes(h-k)]\} \qquad (10)
\end{aligned}$$

Introducing Eqs. (8) and identifying the actual strain rates of the discrete again by Eqs. (9), the mean power of the module can be written in the form

$$\begin{aligned}
\pi(\mathbf{w},\mathbf{W},\mathbf{d}) &= (\mathbb{A}(\mathbf{H}-\mathbf{W})+\mathbb{B}\mathsf{H})\cdot(\mathbf{H}-\mathbf{W})+\frac{1}{2}(\mathbb{C}(\mathbf{H}-\mathbf{W})+\mathbb{D}\mathsf{H})\cdot\mathsf{H} \\
&+ (\mathbb{E}\mathbf{d}+\mathbb{F}\mathbf{N})\cdot\mathbf{d}+(\mathbb{G}\mathbf{d}+\mathbb{H}\mathbf{N}+\mathbb{I}\mathbf{d}^2+\mathbb{L}\mathbf{N}^T\mathbf{d}+\mathbb{M}\mathbf{N}^T\mathbf{N})\cdot\mathbf{N}\,. \qquad (11)
\end{aligned}$$

The components of the sixth order tensor $\mathbb{D}$, the fifth order tensors $\mathbb{B}$, $\mathbb{C}$, the fourth order tensors $\mathbb{A}$, $\mathbb{H}$, $\mathbb{M}$, the third order tensors $\mathbb{F}$, $\mathbb{G}$, $\mathbb{L}$ and the second order tensors $\mathbb{E}$, $\mathbb{I}$, depend on the elastic constants of the matrix and on the size, the orientation, and the arrangement of the internal phases. The non linear terms in (11) account for the interactions between microcracks.

To consider also the interactions between the fibers and the microcracks the hypothesis of rigid bonds between the two lattices should be removed and the power of the internal actions $\mathbf{r}_l$ should be considered.

The equivalent macroscopic model can be identified by requiring that the mean power of the interactions of the module (11) equals the mean density power of a neighbourhood of a continuum for any admissible $\mathbf{H}-\mathbf{W}$, H, $\mathbf{d}$ and $\mathbf{N}$. Tacking into account of the localization theorem, the stress density power of the equivalent continuum can be approximated by the formula

$$den\,\mathcal{P}^{int} = \mathbf{S}\cdot(\mathbf{H}-\mathbf{W})+\frac{1}{2}\mathsf{S}\cdot\mathsf{H}+\mathbf{z}\cdot\mathbf{d}+\mathbf{P}\cdot\mathbf{N}\,, \qquad (12)$$

where the dynamical quantities are

$$\begin{aligned}
\mathbf{S} &= \mathbb{A}(\mathbf{H}-\mathbf{W})+\mathbb{B}\mathsf{H}\,, \qquad & \mathsf{S} &= \mathbb{C}(\mathbf{H}-\mathbf{W})+\mathbb{D}\mathsf{H}\,, \\
\mathbf{z} &= \mathbb{E}\mathbf{d}+\mathbb{F}\mathbf{N}\,, & \mathbf{P} &= \mathbb{G}\mathbf{d}+\mathbb{H}\mathbf{N}+\mathbb{I}\mathbf{d}^2+\mathbb{L}\mathbf{N}^T\mathbf{d}+\mathbb{M}\mathbf{N}^T\mathbf{N}\,. \qquad (13)
\end{aligned}$$

Thus, because of the requirement of power preservation, the lattice system described in Section 2 must be replaced by a structured continuum that admits the fields $\mathbf{H} - \mathbf{W}$, $\mathbf{d}$ and $\mathbf{N}$ as linearized strain measures to each of which a distinct stress measure, identified through Eqs. (13), corresponds. If a continuum characterized by different kinematical descriptors were adopted, it would be impossible to impose the power equivalence unless internal constraints or restrictions on the form of the constitutive tensors were given; in this situation the set of the admissible velocities or the geometry of the lattice system should be modified [11].

4. THE CONTINUUM WITH AFFINE STRUCTURE

The continuum characterized by the strain and stress fields introduced in the above session belongs to the class of the solids endowed with affine microstructure [7].[4] These continua can undergo homogeneous microdeformations independent of the local macroscopic deformation and can be regarded as multifield materials.[5]

Let two three-dimensional smooth manifolds, namely the group $\mathcal{SO}(3)$ of the proper orthogonal transformations and the vector space $\mathcal{V}$ of the translations of $\mathcal{E}$, be associated to the Euclidean region occupied by the multi-phase body B. The triplet

$$x = x(P)\,, \qquad \mathrm{R} = \mathrm{R}(P)\,, \qquad \mathrm{v} = \mathrm{v}(P)\,, \qquad \text{with } P \in \mathrm{B}\,,\ x \in \mathcal{E}\,,\ \mathrm{R} \in \mathcal{SO}(3)\,,\ \mathrm{v} \in \mathcal{V}\,, \tag{14}$$

constitutes a 'complete placement' for B. The microstructural fields $\mathbf{R}$ and $\mathbf{v}$ represent the rigid and the deformable local structure respectively: with an eye to the original discrete model, the former accounts for the presence of very strong and stiff fibers, the latter for the presence of distributed microcracks in the matrix. Due to the mean displacement jumps which would occur over the actual microcracks, the material point P can be seen as occupying the place $x + \mathrm{v}$. By defining for B a reference shape $\mathcal{C}$, the overall displacement of the body is $\mathbf{u}^t = x - X + \mathrm{v}$, where $X \in \mathcal{C}$. Thus, the vector field v represents the difference between $\mathbf{u}^t$ and the displacement of the flawless body. It can also be interpreted as a deformable director associated to the material particle [8]. The following deformation gradients can be then introduced

$$\mathrm{F}^t = \mathrm{F} + \operatorname{grad}\mathrm{v} = \operatorname{grad} x + \mathrm{I} + \operatorname{grad}\mathrm{v}\,, \qquad \mathsf{F} = \operatorname{grad}\mathrm{R}\,, \tag{15}$$

where 'grad' denotes differentiation with respect to X. The additive decomposition in Eqs. (15) has already been used for particular fiber continua [23] and can be shown to correspond to the classical multiplicative decomposition $\mathbf{F}^t = \mathbf{F}\mathbf{F}^m$, where $\mathbf{F}^m$ stands for the deformative contribution due to the presence of microcracks [18]. As strain measures we define[6]

$$\mathrm{U} = \mathbf{R}^T\mathrm{F} - \mathrm{I}\,, \qquad \mathsf{U} = \mathbf{R}^T \circ \mathsf{F}\,, \qquad \mathbf{u} = \mathbf{R}\mathrm{v}\,, \qquad \mathbf{U}^m = \mathbf{R}\operatorname{grad}\mathrm{v}\,. \tag{16}$$

These account for the coupling between the rotation of the fibers and the deformation of the microflaws which, roughly speaking, consists of mean crack-opening displacements and of relative mean displacement jumps between two microcracks.

With reference to the polar decomposition, we say that the multifield continuum undergoes a rigid deformation when the symmetric, positive definite, part of $\mathbf{F}$ coincides with the identity I and when its orthogonal part Q coincides with the microrotation $\mathbf{R}$ and this is constant. In that case

$$\mathrm{U}^* = 0\,, \qquad \mathsf{U}^* = 0\,, \qquad \mathbf{u}^* = \mathbf{Q}\mathrm{v}\,, \qquad \mathbf{U}^{m*} = \mathbf{Q}\operatorname{grad}\mathrm{v}\,, \tag{17}$$

where the suffix '$*$' stands for 'rigid'.

[4]The so-called micromorphic materials [8] are hyperelastic media with kinematical affine microstructure.

[5]This approach is consistent with the approach proposed in [20, 21] and is also in accord with the one in [22].

[6]In terms of components $[\mathbf{R}^T \circ \mathsf{F}]_{ijk} = [\mathbf{R}]_{ih}[\mathsf{F}]_{hjk}$.

Assuming $\mathbf{R}(X) = \mathbf{I}$, $\mathbf{v}(X) = \mathbf{0}$ and by linearizing the expressions (16) near the reference configuration C, we obtain as strain rates

$$\tilde{\mathbf{U}} = \mathbf{H} - \mathbf{W}, \qquad \tilde{\mathsf{U}} = \mathsf{H}, \qquad \tilde{\mathbf{u}} = \mathbf{d}, \qquad \tilde{\mathbf{U}}^m = \mathbf{N}, \tag{18}$$

where $\mathbf{w} = \dot{x}$, $\mathbf{H} = \operatorname{grad}\mathbf{w}$, $\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T$, $\mathsf{H} = \operatorname{grad}\mathbf{W}$, $\mathbf{d} = \dot{\mathbf{v}}$ and $\mathbf{N} = \operatorname{grad}\dot{\mathbf{d}}$: a superimposed dot denotes the derivative with respect to a real parameter ϵ evaluated for $\epsilon = 0$. Consistently with the discrete model of Section 2, at the infinitesimal level, the two microstructures do not interact.

In agreement with the principle that two observers can be made to coincide through a rigid rotation $\mathbf{Q}$, the rigid velocity fields of the microstructured continuum are then characterized by the following equations

$$\mathbf{w}^* = \mathbf{c} + \mathbf{W}^*(x - o), \qquad \mathbf{W}^* = \dot{\mathbf{Q}}\mathbf{Q}^T, \qquad \mathbf{d}^* = \mathbf{W}^*\mathbf{d}, \qquad \mathbf{N}^* = \mathbf{W}^*\mathbf{N}, \tag{19}$$

where the vector $\mathbf{c}$ and $\mathbf{W}^*$ are respectively the translational and the angular velocity.

Let $\mathbf{b}$ and $\mathbf{B}$ be the vector and the skew-symmetric second order tensor of the volume density of the external forces and couples respectively, $\mathbf{f}$ and $\mathbf{M}$ the vector and the the skew-symmetric second order tensor of surface density of the external forces and couples and $\mathbf{p}$ the vector of the surface density of the external actions on the deformable microstructure; the following expression for the power of the external actions can be written

$$\mathcal{P}^{est} = \int_C (\mathbf{b}\cdot\mathbf{w} + \frac{1}{2}\mathbf{B}\cdot\mathbf{W})\,dV + \int_{\partial C} (\mathbf{f}\cdot\mathbf{w} + \frac{1}{2}\mathbf{M}\cdot\mathbf{W}) + \mathbf{p}\cdot\mathbf{d})\,dS. \tag{20}$$

The additional surface forces $\mathbf{p}$ acting on the boundary of the body ∂C, can be assigned through a constitutive prescriptions. For example they are the forces due to displacement jump distributions caused by external tractions on ∂C. No external body forces acting on the microcracks are considered.

Associating a dynamical quantity to each strain descriptors, the internal power has finally the expression required by the adopted procedure of equivalence

$$\mathcal{P}^{int} = \int_C (\mathbf{S}\cdot(\mathbf{H} - \mathbf{W}) + \frac{1}{2}\mathsf{S}\cdot\mathsf{H} + \mathbf{z}\cdot\mathbf{d} + \mathbf{P}\cdot\mathbf{N})dV, \tag{21}$$

where $\mathbf{S}$ represents the macroscopic second order stress tensor and S the third order couple-stress tensor, while $\mathbf{z}$ and $\mathbf{P}$ are respectively the vector of the internal microstructural actions and the microscopic second order stress tensor. The constitutive functions of these stress measures are identified from the discrete model through Eqs. (13); in particular $\mathbf{z}$ and $\mathbf{P}$ represent the additional state of stress on the body due to the presence of microcracks and to their interactions.

According to the axiomatic description of Di Carlo [24] for materials with microstructure, for which the standard invariance under changes of observer is not sufficient to derive the balance of the microactions [25], let us require $\mathcal{P}^e = \mathcal{P}^i$ for any $\mathbf{H} - \mathbf{W}$, H, $\mathbf{d}$ and $\mathbf{N}$ and for any subset $\bar{C} \subseteq C$: then, using a standard procedure, the balance equations of forces and microforces and the balance equation of couples on C can be obtained

$$\operatorname{div}\mathbf{S} + \mathbf{b} = \mathbf{0}, \qquad \operatorname{div}\mathbf{P} - \mathbf{z} = \mathbf{0}, \qquad \operatorname{div}\mathsf{S} + \mathbf{S}^T - \mathbf{S} + \mathbf{B} = \mathbf{0} \tag{22}$$

while on ∂C

$$\mathbf{S}\mathbf{n} = \mathbf{f}, \qquad \mathbf{P}\mathbf{n} = \mathbf{p}, \qquad \mathsf{S}\mathbf{n} = \mathbf{M}. \tag{23}$$

If only the rigid structure were present, the inner density power would be zero for any rigid velocity field, and no equation must be added to Eqs. (22). On the contrary, in the case studied here, not all the strain measures vanish in a rigid déformation but the power of the corresponding stresses must be zero for any $\bar{C} \subseteq C$. Therefore, by requiring the internal density power be null for any rigid velocity field (Eqs. 19), the following restrictions on the microstresses must be imposed

$$\mathbf{z}\otimes\mathbf{d} - \mathbf{d}\otimes\mathbf{z} + \mathbf{P}\mathbf{N}^T - \mathbf{N}\mathbf{P}^T = \mathbf{0}. \tag{24}$$

This equation is essentially a constitutive prescription on the stress fields.[7] It can be noted that, due to the presence of the rigid structure, the skew-symmetric part of the stress tensor S must satisfy the balance equation (22c), while no constitutive prescriptions are given on it, even if the material is constrained to have the velocity of microrotation equal to the velocity of macrorotation, $2\mathbf{W} = \mathbf{H} - \mathbf{H}^T$.Conversely, if the rigid structure were absent, the condition of zero internal density power for virtual rigid velocity distributions would write

$$den\mathcal{P}^{int} = \mathbf{S} \cdot \mathbf{W}^{\cdot} + \mathbf{z} \cdot \mathbf{W}^{\cdot}\mathbf{d} + \mathbf{P} \cdot \mathbf{W}^{\cdot}\mathbf{N}, \tag{25}$$

and Eq. (24) would become the standard equation of continua with vectorial microstructure

$$\mathbf{S} - \mathbf{S}^T + \mathbf{z} \otimes \mathbf{d} - \mathbf{d} \otimes \mathbf{z} + \mathbf{P}\mathbf{N}^T - \mathbf{N}\mathbf{P}^T = \mathbf{0}, \tag{26}$$

which would require the symmetry of S in absence of microstructure.

5. MATERIAL SYMMETRIES

The constitutive parameters of the equivalent continuum are identified when the elastic constants of the matrix and the geometry of its internal phases are known. To describe the arrangement of the original lattice system, the knowledge of the the material symmetries of the body can be taken into account. Here below we show that, as a trivial consequence of the procedure of power equivalence, the equivalent continuum preserves these symmetries.

Any orthogonal transformation Q, acting either on the reference shape of the module or on its deformation, and leaving unchanged the power π is a symmetry transformation for the lattice system. Indicating with $\mathbf{Q}*$ the action of Q on an element of a vector field,[8] the symmetry group of the discrete can be defined as the group of the orthogonal transformations for which

$$\pi(\mathbf{w}_i, \mathbf{W}_i, \mathbf{d}^h, \mathbf{d}_j) = \pi(\mathbf{Q} * \mathbf{w}_i, \mathbf{Q} * \mathbf{W}_i, \mathbf{Q} * \mathbf{d}^h, \mathbf{Q} * \mathbf{d}_j) \tag{27}$$

in any allowable strain measure.[9]

Limiting by definition allowable strains to those expressed as functions of the smooth fields w, W, d and N, the material symmetry group can be represented as follows

$$\begin{aligned} \mathcal{G} = \{\mathbf{Q} \in \mathcal{O}(3) \quad | \quad & \mathbf{Q} * (\mathbb{A}(\mathbf{H} - \mathbf{W}) + \mathbb{B}\mathbf{H}) = \mathbb{A}\mathbf{Q} * (\mathbf{H} - \mathbf{W}) + \mathbb{B}\mathbf{Q} * \mathbf{H} \\ & \mathbf{Q} * (\mathbb{C}(\mathbf{H} - \mathbf{W}) + \mathbb{D}\mathbf{H}) = \mathbb{C}\mathbf{Q} * (\mathbf{H} - \mathbf{W}) + \mathbb{D}\mathbf{Q} * \mathbf{H} \\ & \mathbf{Q} * (\mathbb{E}\mathbf{d} + \mathbb{F}\mathbf{N}) = \mathbb{E}\mathbf{Q} * \mathbf{d} + \mathbb{F}\mathbf{Q} * \mathbf{N} \\ & \mathbf{Q} * (\mathbb{G}\mathbf{d} + \mathbb{H}\mathbf{N} + \mathbb{I}\mathbf{d}^2 + \mathbb{L}\mathbf{N}^T\mathbf{d} + \mathbb{M}\mathbf{N}^T\mathbf{N}) = \\ & \mathbb{G}\mathbf{Q} * \mathbf{d} + \mathbb{H}\mathbf{Q} * \mathbf{N} + \mathbb{I}\mathbf{Q} * \mathbf{d}^2 + \mathbb{L}\mathbf{Q} * (\mathbf{N}^T\mathbf{d}) + \mathbb{M}\mathbf{Q} * (\mathbf{N}^T\mathbf{N}) \,, \end{aligned} \tag{28}$$

where, since we are dealing with tensors of any (even and odd) order, the full orthogonal group $\mathcal{O}(3)$ has been considered. All the terms of the power have been uniquely identified by Eqs. (13); hence discrete and continuum materials share the same symmetry group. It is worth noting finally that by introducing internal constraints on a structured continuum, a continuum with latent microstructure [7] equivalent to the same discrete system can be derived. In this case the symmetry group of the original material is a subgroup of the symmetry group of the constrained material [17].

[7] Di Carlo [24] pointed out the difference between balance equations obtained as Eqs. (22), which must be regarded as selection rules on processes, and equations obtained as Eqs. (24), representing selection rules on constitutive prescriptions.

[8] In terms of components $[\mathbf{Q} * \mathbf{T}]_{ab...n} = [\mathbf{T}]_{\alpha\beta...\nu}[\mathbf{Q}]_{a\alpha}[\mathbf{Q}]_{b\beta}...[\mathbf{Q}]_{n\nu}$.

[9] This definition accounts for both the material and the geometrical symmetries of the module and agrees with the definition by Noll [26] of the isotropy group of simple solid materials.

6. CONCLUDING REMARKS

The continuum model proposed for composite materials accounts for the presence of inclusions and microflaws and for their spatial correlations by means of additional kinematical and dynamical descriptors. The constitutive parameters include the material length scales of the two internal phases. In further work we shall try and apply the present model to tackle concrete problems. We feel encouraged by the results obtained separately for masonry-like materials described as Cosserat continua [17], which showed the possibility to treat problems with load or geometrical singularities. and for microcracked bodies [18], which pointed out the the effect of the vectorial microstructure in a one-dimensional damaged structure. Moreover, because of the structure of the field equations (more partial differential equation with the gradients of the microstructural fields as field variables), we expect to be able to solve problems of stress or strain concentrations in which the characteristic dimension is comparable to the internal length scales.

Acknowledgments. This work has been partially supported by the 'Ministero dell'Università e della Ricerca Scientifica' under a grant MURST 60%96.

References

[1] Hashin Z., *J. Appl. Mech.*, **50**, (1983), 481–505.
[2] Boutin C., *Int. J. Solids Structures*, **33**, (1996),1023–1051.
[3] Beran M. J., McCoy J. J., *Int. J. Solids Structures*, **6**, (1970), 1035–1054.
[4] Gurtin M. E., *Arch. Rational Mech. Anal.*, 19, (1965), 339–352.
[5] Dunn J. E., Serrin J., *Arch. Rational Mech. Anal.*, (1984), 95–133.
[6] Capriz G., *Arch. Rational Mech. Anal.*, **90**, (1985), 43-56.
[7] Capriz G., Continua with Microstructure, (Springer-Verlag, 1989).
[8] Eringen A. C., "Mechanics of micromorphic continua" (Proc. IUTAM Symp., Freudenstadt-Stuttgart, 1967) E. Kröner Ed., Springer-Verlag, Berlin, (1968) pp. 18-35.
[9] Capriz G., Podio Guidugli P., *Ann. Mat. Pura Appl.*, **IV**, (1976), 195–211.
[10] Ericksen J. L., "Special topics in elastostatics", C. S. Yih Ed. (Advances in Applied Mechanics. Academic Press, New York, 1977) pp. 189–244.
[11] Trovalusci P., Masiani R., (1998), *Int. J. Solids Structures.* In print.
[12] Besdo D., *Arch. Mech.*, **37**, (1985), 603–619.
[13] Mühlhaus H. B., *Ing. Arch.*, **59**, (1989), 124–137.
[14] Dawson E. M., Cundall P. A., Slope stability using micropolar elasticity, Hassani, Aubertin and Mitri Eds. (Rock Mechanics, Balkema, 1996) pp. 551–558.
[15] Lemaitre J., A course on damage mechanics. (Springer-Verlag, Berlin, 1992).
[16] Masiani R., Rizzi N., Trovalusci P., *Meccanica*, 30, (1995), 673–683.
[17] Masiani R., Trovalusci P., *Meccanica*, **31**, (1996), 421–432.
[18] Mariano P. M., Trovalusci P., *Int. J. Damage Mech.*, (1998). In print.
[19] Landau L. D., Lifšit E. M., Teoria dell'elasticità (transl. from Russian, Editori Riuniti, Rome. 1972).
[20] Augusti G., Mariano P. M., *Prob. Eng. Mech.*, 11, (1996), 107–112.
[21] Mariano P. M., Augusti G., *Math. Mech. Solids*,(1998). In print.
[22] Frémond M., Nedjar B., *Int. J. Solids Structures*, **33**, (1996), 1083–1103.
[23] Saczuk J., *Rendiconti Mat.*, **16**, (1996), 315–327.
[24] Di Carlo. A., "A non-standard format for continuum mechanics", Contemporary research in the mechanics and mathematics of materials, R. C. Batra and M. F. Beatty. Eds. (CIMNE. Barcelona, 1996) pp. 92-104.
[25] Gurtin E., Podio Guidugli P., *Z. angew Math. Phis.*, **43**, (1992),181–190.
[26] Noll W., *Arch. Rational Mech. Anal.*, **2**, (1958), 198–226.