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Constitutive Relations for Elastic Microcracked Bodies: From a Lattice Model to a Multifield Continuum Description

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ABSTRACT: A continuum model suitable for the description of microcracked bodies is shown. The influence of microcracks on the mechanical behavior of the body is estimated through a microstructural field added to the displacement one. This field represents the perturbation to the regular displacement field due to the presence of microcracks. It is an observable quantity; its rate must satisfy appropriate balance equations. The problem of deriving constitutive relations for such a model at least in the linear elastic case is dealt with. Constitutive equations are deduced from a lattice model using an integral identification procedure based on the equivalence in terms of virtual work, without resorting to limit processes. The discrete model considered is made of two superposed lattices: the first one is constituted of material points connected by elastic links; the second one is made of empty closed shells interacting between themselves and with the first lattice. As sample test, a one-dimensional problem is shown.

1. INTRODUCTION

THE PRESENT PAPER deals with the deduction of the constitutive equations of a continuum model of microcracked bodies from a lattice frame. Such a deduction is made through an identification procedure based on the equivalence of the virtual work of the two models. The constitutive equations of the continuum are derived in terms of the interaction forces in the discrete model.

The continuum model is based on a scheme of a continuum with structure (Capriz, 1989). In particular, in the present paper, in addition to the displacement

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field, a kinematical microstructural field is considered. It represents the perturbation of the displacement field due to the presence of microcracks.

The point of view of microstructured continua is not in opposition to the internal variable one, classically used in the damage mechanics field (e.g., see Lubarda and Krajcinovic, 1995; Budianski and O'Connell, 1976; Krajcinovic, Basista, and Sumarac, 1991; Bazant and Pijauder-Cabot, 1988; Lemaitre, 1992; Krajcinovic, 1996). In both approaches, additional information is ascribed to the material particle besides its geometrical position. The internal variable models differ from the microstructural ones because the additional variables (say order parameters) are not observable variables, and they need not satisfy principia common to the observable ones. Their presence is noticed only by means of their time evolution rule, which is usually assumed in an empirical way. In the multifield (microstructural) description, the additional field is observable, and its rate must satisfy balance equations (with classical physical meaning). These equations involve internal and external actions on the microstructure.

From a mathematical point of view, a body with additional descriptors, either internal variables or microstructural fields, can be considered as a fiber submanifold of a fibered environmental space. The existence of a mapping that associates to each point of the body an element of a finite-dimensional differentiable manifold is assumed. In the fiber bundle description, each fiber is this finite-dimensional manifold. Two different fibers are assumed in the present context to be copies of the same manifold. In such a way, depending on the choice of the kind of order parameters, microstructured continua schemes or internal variables are respectively obtained if an intrinsic connection[2] on the fiber manifold can or cannot be defined. Without such a connection there is no possibility to observe the order parameters, and their time evolution is only given by kinetic rules; hence, they are in essence internal variables (see the detailed discussions in Capriz and Giovine, 1997; Di Carlo, 1996; Segev, 1994).

Besides the usually noticed problems of ill-posedness in balance equations, arising when the strain-softening occurs, and mesh dependence of numerical schemes, another problem of the internal variable schemes is associated to the application of a theorem by Morro (1982) to damage models. The theorem states that the asymptotic stability of internal variable models implies the independence of the internal variable from the actual values of the observable ones. This fact leads to some questionable implications. For example, the damage variables are independent from the actual value of the stress.

In the writers' opinion, despite some computational burden due to the increase of the field equations, the microstructural approach should bypass some of these

[2] A connection is a way to mathematically describe the remote parallelism between vectors. For the analytical definition, see a generic textbook of differential geometry (e.g., Warner, 1983)

difficulties (Augusti, and Mariano, 1996; Mariano, 1995, 1996; Mariano and Augusti, 1997; Markov, 1996; Frémond and Nedjar, 1996); because of the property of its slightly different mathematical structure, the microstructural fields must satisfy balance equations which are partial differential equations, differently from the internal variable models. Moreover, from a numerical point of view, a mesh independence is expected because the continuum structure involves the gradient of the additional field (e.g., Steinmann, 1995, in which a damaged Cosserat continuum is considered, and the damage is described by internal variables).

A basic equation in the microstructural approach is the identification of the constitutive equations for the macroscopic and microscopic stress fields. In the present paper, such an identification is inferred from a discrete description.

Generally, a lattice model provides a detailed description of the local behavior of the material by using simple constitutive relations for the interactions between the elements of the system. A procedure of equivalence proposed by Masiani, Rizzi, and Trovalusci (1995) is here used to bypass some difficulties concerning the transition from a discrete to a continuum description. This procedure has been developed for rigid block systems regarded as micropolar continua, and its utility in structural applications has been shown in Masiani and Trovalusci (1996).

The equivalence between the two models is based on the requirement that the work of the internal actions in the lattice equals the power of the continuum for any compatible virtual field. The equivalence is also based on the assumption that the strain measures of the discrete model must correspond, through a suitable map, to the strain field of the continuous model. In such a way, all the terms of the mechanical power can be identified and explicit constitutive functions for the continuum stress measures can be obtained without resorting, as in some homogenization techniques, to limiting processes.

A complex lattice model is employed to identify the constitutive functions of the multifield continuum. This lattice is represented by two networks: the first, representing the matrix, consists of material particles connected by linear elastic bonds; the second is made of empty closed shells, representing the microcracks, connected to each other by elastic links and connected with the first lattice by rigid bonds.

As an example, the solution of a one-dimensional case is shown in detail.

2. CONTINUUM MODEL

From a geometrical point of view, a microcracked body can be identified with a region, B, of the Euclidean three-dimensional space, obtained by a finite union of piecewise fit regions. A fit region is a bounded, regularly open set with a finite perimeter of zero volume. It is assumed that the microcracks are distributed on B. In that region, there exists an unknown set, $\mathcal{G}_m$, on which the displacement field is discontinuous and such that its derivative in the sense of distributions[3] is given by

$$D\mathbf{u} = \nabla \mathbf{u} \cdot L^n(\mathcal{B}) + [\mathbf{u}] \otimes \mathbf{n}_{\mathcal{G}_m} \cdot \mathcal{H}^{n-1}(\mathcal{B}) + C\mathbf{u} \tag{1}$$

where L^n is the n-dimensional Lebesgue measure,[4] [u] is the displacement jump over the microcracks. $\mathcal{H}^{n-1}$ is the Hausdorff $n-1$ dimensional measure of the reduced boundary[5] of the defects set, and $C\mathbf{u}$ is the Cantor measure, while $\mathbf{n}_{\mathcal{G}_m}$ is the outward normal to the reduced boundary of $\mathcal{G}_m$. If point defects are absent, the Cantor measure is zero. Although an approach based on Equation (1) could be appealing, it presents considerable analytical difficulties and is only employed for the study of static processes of elastic microcracked bodies,[6] at least at the present state of technical literature.

Alternatively, here below the microcracked body is represented by a continuum endowed with a local vector structure. The microcracks are assumed to be uniformly distributed on the whole body. The body occupies a fit region, $\mathcal{B}$, of the Euclidean three-dimensional space. A smooth manifold is also assigned. This manifold is a three-dimensional vector space. Its elements represent the various microstructural conditions. To each material particle, P, a position $\mathbf{x}$ and a vector d are associated. The pair $(\mathbf{x}(P), d(P))$ is called complete placement. From a mathematical point of view, this approach can be regarded as considering the body like a fiber bundle in which each point of the base represents the position of the center of mass of a material particle in the Euclidean space, while each fiber contains more or less rough information about the texture of that particle, namely about its microcracks in the present case.

The microstructural field d, is the perturbation induced on the regular macroscopic displacement field by the mean displacement jump which it would have over the actual microcrack at P. It can be interpreted as the difference between the actual displacement field in the body with cracks and the regular displacement field of the same body without cracks.[7] As it is assumed that d represents a homogeneous deformation, the continuum belongs to the class of continua with affine structure.[8] Such a continuum is then described by two independent vector fields: the micro-displacement field **u** and the micro-displacement field d.

[3]This point of view departs from the basic general ideas formulated by De Giorgi (1993) for problems with free discontinuities, developed by Ambrosio (1989, 1990) and recently applied to cracked continua by Braides, Defranceschi, and Vitali (1996) for the study of static process in which the growth of microcracks is regulated by the Griffith's criterion.
[4]The value of n is not specified because the body can be three-, two-, or one-dimensional.
[5]The reduced boundary of a set is the part of the boundary on which the outward normal can be defined uniquely.
[6]In this case, the study of a body in elastic phase with the geometry described above can be reconduced to find the minimum of the energy written in a way such to consider discontinuities. This variational problem has two unknown quantities: the displacement field, **u**, and the set of discontinuities, $\mathcal{G}_m$. The existence, the uniqueness, and the asymptotic behaviour of it have been studied recently (Braides, Defranceschi, and Vitali, 1996).
[7]The field d can be considered as a regularized distribution and can be interpreted as the difference between the real transplacement, obtained as limit in L^∞ of displacements that induces simple deformations and the regular one that the body would have if cracks were absent (see Del Piero and Owen, 1993, 1995, for detailed theory). The L^∞ topology is necessary to map bodies into bodies.
[8]With reference to the theory of micromorphic materials (Mindlin, 1964; Capriz, 1989; Grioli, 1990), d can also be interpreted as a field of deformable directors associated to the material particles.

In this framework, two independent deformation gradients are considered, namely,

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u} \quad \text{and} \quad \mathcal{N} = \nabla d \tag{2}$$

where **I** is the identity tensor.

An overall deformation gradient can be considered in two different ways. An additive decomposition[9]

$$\mathbf{F}_t = \mathbf{I} + \nabla\mathbf{u} + \mathcal{N} \tag{3}$$

corresponds, in a certain sense, to assume the body with structure embedded into a particular Finslerian fiber bundle.[10] Some interesting aspects of this assumption can be found in Saczuk (1996). In that case, the left Cauchy-Green strain tensor is

$$\mathbf{C} = \mathbf{I} + \nabla\mathbf{u} + \nabla\mathbf{u}^T + \mathcal{N} + \mathcal{N}^T + \nabla\mathbf{u}^T\nabla\mathbf{u} + \nabla\mathbf{u}^T\mathcal{N} + \mathcal{N}^T\nabla\mathbf{u} + \mathcal{N}^T\mathcal{N} \tag{4}$$

and the overall strain tensor

$$\mathbf{E}_t = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \tag{5}$$

can be introduced.

To linearize it, consider the real numbers given by

$$\varepsilon_1 = |\nabla\mathbf{u}|, \quad \varepsilon_2 = |\mathcal{N}|, \quad \varepsilon^2 = \min\{\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1\varepsilon_2\} \tag{6}$$

As a consequence

$$\mathbf{C} = \mathbf{I} + 2\mathbf{E}^o + 2\mathbf{E}^m + o(\varepsilon^2) \tag{7}$$

where

$$\mathbf{E}^o = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \quad \text{and} \quad \mathbf{E}^m = \frac{1}{2}(\mathcal{N} + \mathcal{N}^T) \tag{8}$$

The overall linearized strain tensor becomes

$$\tilde{\mathbf{E}}_t = \mathbf{E}^o + \mathbf{E}^m \tag{9}$$

[9]Note that in the framework of elasto-plastic materials, the additive decomposition (6) is consistent with the natural multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, if the elements of the decomposition are defined as $\mathbf{F}^e = (\mathbf{I} + \nabla\mathbf{u} + \mathcal{N})^e$ and $\mathbf{F}^p = (\mathbf{I} + \nabla\mathbf{u} + \mathcal{N})^p$.
[10]Such an idea corresponds to consider the body as a fiber bundle in which the metric is a convex function of **u** and d and the remote parallelism is described by a Cartan's connection (Rund, 1959).

where the constants have the following values: ($\mathcal{A} = E = 10^5$ Kg\cm^2; $\mathcal{B} = E/\pi l l_c = 3.183 \cdot 10^5$ Kg\cm^4; $C = D = 0$ Kg\cm^3; $\mathcal{E} = 3.915\,(E/\pi l_c)l = 1.246 \cdot 10^8$ Kg\cm^2; $\mathcal{F} \rightarrow 10^{-2}$ Kg\cm^4; $\mathcal{G} \rightarrow -10^{-2}$ Kg\cm^3; $\mathcal{H} \rightarrow 10^{-2}$ Kg\cm^2; the length of the module $l = 10$ cm, $l_c = 10$ cm. The microcrack density per unit length (MDUL) of the one-dimensional lattice considered is equal to 4.8. The equilibrium equations are written as follows:

$$\begin{cases} \mathcal{A}u'' = 0 \\ \mathcal{E}d'' - \mathcal{B}d + (\mathcal{D} - C)d' + \mathcal{G}d'^2 + \mathcal{G}dd'' + 2\mathcal{F}dd' + 2\mathcal{H}d'd'' = 0 \end{cases} \tag{52}$$

For the sake of simplicity, only the linear part of the constitutive equations is considered. As a consequence, the balance equations assume the following form:

$$\begin{cases} \mathcal{A}u'' = 0 \\ \mathcal{E}d'' - \mathcal{B}d + (\mathcal{D} - C)d' = 0 \end{cases} \tag{53}$$

Note that, since the material is hyperelastic, the constants $\mathcal{E}$ and $\mathcal{B}$ are positive and the second balance equation in formula (53) has real eigenvalues.

Under the initial conditions

$$\begin{cases} u(0) = 0 \\ u'(0) = \dfrac{F}{EA} \end{cases} \quad \text{and} \quad \begin{cases} d(0) = \delta \\ d'(0) = 0 \end{cases} \tag{54}$$

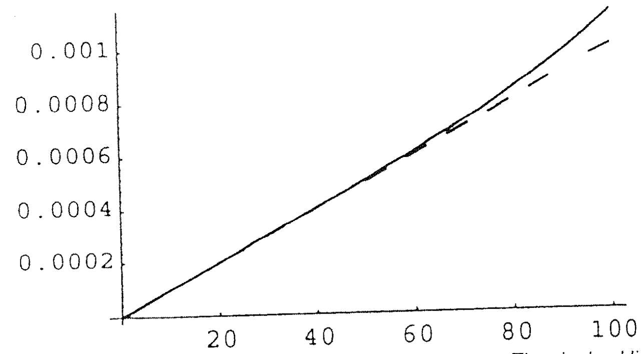

Figure 7. *Displacement diagram:* **u** + *d versus the coordinate* **x**. *The dashed line represents the diagram of* **u**.

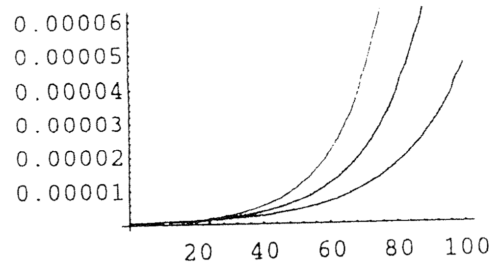

Figure 8. *Different diagrams of the displacement perturbation d versus the coordinate* **x**, *varying α for different microcracks densities. The following values of α are considered: $\alpha = 0.07$ (upper; MDUL = 6.714); $\alpha = 0.06$ (medium; MDUL = 5.875); $\alpha = 0.05$ (lower; MDUL = 4.8).*

where $\delta = \pi F L_c / \hat{A}E = 10^{-7}\,\pi$, the solution of the problem (53)–(54) is given by

$$\begin{cases} u(x) = \dfrac{F}{EA}x \\ d(x) = \dfrac{\delta}{2}(e^{\alpha x} + e^{-\alpha x}) \end{cases} \tag{55}$$

The value of α is 0.05. A detailed picture is given in Figures 7 and 8.

The perturbation given by the microdisplacement field d is nonlinear. The obtained nonlinearity of the microstructural field d is due to weak nonlocal effects of microcracks that are intrinsic into the gradient of d in the constitutive equations (51). If an effective modulus method were used to solve the same problem, the result would be a linear diagram of the displacements with an elastic modulus different from the one of the virgin material.

5. CONCLUDING REMARKS

A continuum multifield description for microcracked elastic bodies equivalent to a lattice model has been proposed. The adopted multifield continuum model directly accounts for the presence and the spatial correlation between microcracks by means of additional kinematical and dynamical descriptors. This approach circumvents some theoretical problems related to the classical models with internal variables. The proposed model entails some difficulties related to the definition of the constitutive functions of all the dynamic quantities. Such problems have been tackled by an identification procedure based on an integral criterion of equivalence. The constitutive equivalent model is completely defined when the elastic constants and the geometry of the lattice model are known.

To select the geometry of the lattice model it is necessary to know the initial distribution of microcracks. Such a distribution can be deduced experimentally (Lubarda and Krajcinovic, 1993). To avoid a large computational burden and to define a representative module on which the identification procedure can be applied, the material must be considered at least statistically homogeneous.

Within the theoretical framework discussed, the model can be extended to describe nonconservative evolution of microcracks by introducing an entropy flux which takes into account the mechanical dissipation due to the opening of microcracks. In such a way, an irreversible kinetics of the microcracks can be obtained by requiring the maximum of the dissipation (Mariano and Augusti, 1997).

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