A constitutive model for fibre composite materials based on microscopic descriptions

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ABSTRACT: Constitutive equations for composite materials with reinforcing fibres and micro-flaws are derived in the framework of continua with microstructure (multi-field continua). These equations are based on the kinematics and the statics of the material at the microscopic level and account for the geometry and the texture of the material internal phases. Considering homogeneous deformations for the microscopic model (lattice model) the equivalent macroscopic model proves to be a multi-field continuum. Differently from the classical continuous models, this continuum includes internal scale parameters which allow taking into account size effects. The study of a one-dimensional problem points out the influence of the microstructure on the macroscopic behaviour of the composite material.

1. INTRODUCTION

At the macroscopic level composite materials, made of short, stiff and strong fibres embedded in a more deformable matrix, present mechanical behaviour that strongly depends on the shape, the size, the orientation and the disposition of the fibres. Moreover, due to manufacturing defects or lack of cohesion, microflaws can be present in the matrix.

In this work, a constitutive model with microstructure is proposed. This model is a continuum with three different material levels: the macrostructural level, the matrix, and two microstructural levels of the internal phases, the fibres and the flaws. In particular, the fibres are perceived as rigid inclusions and the flaws as slit microcracks, both uniformly distributed in the material. Three homogeneous fields describe the kinematics of this continuum: the standard displacement vector field, the micro-rotation tensor field, and a micro-displacement vector field. The microscopic fields represent respectively the rotations of the individual fibres and the distributed displacement jump due to the presence of microflaws in the matrix.

Traditionally, most of the continuous constitutive models proposed, accounting for the wide variety of geometry of the material internal phases as well as of their physical properties, are Cauchy continua developed within the framework of homogenisation theories (Hashin, 1989; Boutin, 1996). These continua present various disadvantages. For instance, due to the absence of internal length scales and of proper kinematical descriptors, they cannot represent the different behaviour of materials made of particles of different size. Moreover, various difficulties arise, when a characteristic dimension of a body is comparable to the internal length scales, like in the well-known strain-localisation phenomena in brittle materials (Read & Hegemier, 1984). Finally, if a thermo-mechanical process is considered, the constitutive assumption of a Cauchy material, even when improved with non-locality (Boutin, 1996), is incompatible with the second law of thermodynamics (Gurtin, 1965).

The multi-field modelling proposed in this work, characterised by the presence of additional kinematical and dynamical field descriptors for the material microstructure, allows to grossly account for the features related to the actual discontinuous and heterogeneous nature of the composite material. This is a way to retain memory of the fine organisation of the material preserving the advantages of the continuum modelling but avoiding the above mentioned difficulties.

2. MICROSCOPIC MODEL

At the microscopic level the particle composite is characterised by distributions of fibres, much stiffer than the matrix in which they are embedded, and of microcracks, which are considered open, stationary and with blunt edges. For the fine description of such materials at this level a discrete model is proposed. This model is made of two interacting lattice systems: a lattice made of rigid particles of

given shape connected in pair by linear elastic beams, representing the matrix with the fibres, and another
lattice made of interacting slits of arbitrary shape with a predominant dimension, representing the microcracks.

- the interaction forces between the Let consider a particle, *A*, interacting with a particle, *B*, through the i-th beam and a slit, *H*, interacting with a slit, *K*, along the j-th direction. Each particle also interacts with the slits placed along the i-th direction.

The linearised strain measures of the assembly are defined as follows:

- the relative displacement between two points, $\mathbf{p}_i^a$ and $\mathbf{p}_i^b$ respectively on *A* and *B*, of the i-th beam connecting the two particles; represented by the vector

$$\mathbf{u}_i = \mathbf{u}_i^a - \mathbf{u}_i^b + \mathbf{R}_i^a(\mathbf{p}_i^a - \mathbf{a}) - \mathbf{R}_i^b(\mathbf{p}_i^b - \mathbf{b}), \qquad (1)$$

where $\mathbf{u}_i^a$ and $\mathbf{u}_i^b$ are the displacement vectors of the centres **a** and **b** of the two particles while $\mathbf{R}_i^a$ and $\mathbf{R}_i^b$ are the skew-symmetric tensors of the rotations of the two particles;

- the relative rotation between the two particles, represented by the skew-symmetric tensor $\mathbf{R}_i = \mathbf{R}_i^a - \mathbf{R}_i^b$;
- the displacement jump on a slit *H*, represented by the vector $\mathbf{d}^h$;
- the relative displacement jump between two slits, *H* and *K*, represented by the vector $\mathbf{d}_j = \mathbf{d}_j^h - \mathbf{d}_j^k$, where $\mathbf{d}_j^h$ and $\mathbf{d}_j^k$ are the displacement jumps of the two slits interacting in the j-th direction**;**
- the relative displacement between the centres, **a** and **h**, of the l-th particle-slit pair, *A-H*, $\mathbf{v}_l = a_l(b_l\mathbf{u}_l + \mathbf{d}_l^h)$, where a_l and b_l are coefficients depending on the number and the position of slits along the l-th direction.

Correspondingly, the internal actions of the lattice systems are:

- the interaction forces between the particles, *A* and *B*, through the i-th beam, represented by the vectors $\mathbf{t}_i^a$ and $\mathbf{t}_i^b$;
- the interaction couples between *A* and *B*, represented by the skew-symmetric tensors, $\mathbf{C}_i^a$ and $\mathbf{C}_i^b$.
- the force due to the displacement jump on a slit *H*, $\mathbf{d}^h$, represented by the vector $\mathbf{z}_0^h$;
- the interaction forces between the l-th particle-slit pair *A-H*, represented by the vectors $\mathbf{r}_l^a$ and $\mathbf{r}_l^h$.

Accounting for the interactions balance and putting

$$\mathbf{t}_i^a = -\mathbf{t}_i^b = \mathbf{t}_i,$$
$$\mathbf{C}_i^a = -\mathbf{C}_i^b - [\mathbf{t}_i^b \otimes (\mathbf{p}_i^a - \mathbf{p}_i^b) - (\mathbf{p}_i^a - \mathbf{p}_i^b) \otimes \mathbf{t}_i^b] = \mathbf{C}_i,$$
$$\mathbf{z}_j^h = -\mathbf{z}_j^k = \mathbf{z}_j,$$
$$\mathbf{r}_l^a = -\mathbf{r}_l^h = \mathbf{r}_l, \qquad (2)$$

the mean work of a representative element of volume V of the discrete system (*lattice module*) is

$$P = \frac{1}{V}\sum_{i=1}^{N}\left\{\mathbf{t}_i \bullet \left[\mathbf{u}_i - \mathbf{R}_i^b(\mathbf{p}_i^a - \mathbf{p}_i^b)\right] + \frac{1}{2}\mathbf{C}_i \bullet \mathbf{R}_i\right\} + \sum_{h=1}^{M}\mathbf{z}_0^h \bullet \mathbf{d}^h + \sum_{j=1}^{K}\mathbf{z}_j \bullet \mathbf{d}_j + \sum_{l=1}^{L}\mathbf{r}_l \bullet \mathbf{v}_l \qquad (3)$$

where N is the number of the beams, M the number of the slits, K the number of the pairs of interacting slits and L the number of the pairs of interacting particles and slits of the module.

2. MACROSCOPIC MODEL

According to the molecular theory of elasticity (Ericksen, 1977), the macroscopic model of the composite material considered is a continuum built up based on the kinematics of proper lattice models (Trovalusci & Masiani, 1999, 2003, Trovalusci & Augusti, 1998). By assuming that the kinematical variables $\mathbf{u}_i^a$, $\mathbf{R}_i^a$ and $\mathbf{d}^h$, for each particle *A* and each slit *H*, are homogeneous, in order of accounting for short-range interactions, the mean work of the lattice module (3) can be expressed in terms of three homogeneous kinematical fields **u**(**x**), **R**(**x**), **d**(**x**)

$$P(\mathbf{u},\mathbf{R},\mathbf{d}) = \mathbf{S}_D \bullet (\nabla\mathbf{u} - \mathbf{R}) + \frac{1}{2}\mathsf{S}_D \bullet \nabla\mathbf{R} + \mathbf{z}_D \bullet \mathbf{d} + \mathbf{P}_D \bullet \nabla\mathbf{d} \qquad (4)$$

where the statical variables

$$\mathbf{S}_D = \frac{1}{V}\{\sum_{i=1}^{N}\mathbf{t}_i \otimes (\mathbf{a}-\mathbf{b}) + \sum_{l=1}^{L}a_l b_l \mathbf{r}_l \otimes (\mathbf{a}-\mathbf{b})\}, \qquad (5)$$

$$\mathsf{S}_D = \frac{1}{V}\sum_{i=1}^{N}\{2\mathbf{t}_i \otimes [(\mathbf{p}_i^a - \mathbf{a}) \otimes (\mathbf{a}-\mathbf{x}) - (\mathbf{p}_i^b - \mathbf{b}) \otimes (\mathbf{b}-\mathbf{x})] + \mathbf{C}_i \otimes (\mathbf{a}-\mathbf{b})\} \qquad (6)$$

$$\mathbf{z}_D = \frac{1}{V}\{\sum_{h=1}^{M}\mathbf{z}_0^h + \sum_{l=1}^{L}a_l\mathbf{r}_l\} \qquad (7)$$

$$\mathbf{P}_D = \frac{1}{\mathrm{V}}\{\sum_{\mathrm{j}=1}^{\mathrm{K}} \mathbf{z}_{\mathrm{j}} \otimes (\mathbf{h}-\mathbf{k}) + \sum_{h=1}^{\mathrm{M}} \mathbf{z}_0^h \otimes (\mathbf{h}-\mathbf{x}) + \sum_{\mathrm{l}=1}^{\mathrm{L}} a_{\mathrm{l}} \mathbf{r}_{\mathrm{l}} \otimes (\mathbf{h}-\mathbf{x})\} \tag{8}$$

depend on the interactions of the lattice systems as well as on the geometry of the internal phases.
Considering the internal density work of a continuum with affine microstructure (Capriz, 1989), with the three homogeneous fields $\mathbf{u}(x)$, $\mathbf{R}(x)$ and $\mathbf{d}(x)$ as kinematical descriptors,

$$W = \mathbf{S} \cdot \nabla\mathbf{u} + \frac{1}{2}\mathsf{S} \bullet \nabla\mathbf{R} + \mathbf{z} \cdot \mathbf{d} + \mathbf{P} \cdot \nabla\mathbf{d} \tag{9}$$

and requiring the equivalence between W and the mean work of the lattice module (4), in any $\mathbf{u}$, $\mathbf{R}$ and $\mathbf{d}$, the macro-stress measures of the equivalent multi-field continuum can be identified as $\mathsf{S}=\mathsf{S}_D$ and the micro-stress measures as $\mathsf{S}=\mathsf{S}_D$, $\mathbf{z}=\mathbf{z}_D$ and $\mathbf{P}=\mathbf{P}_D$. The second order tensor $\mathbf{S}$ is the stress tensor, the third order tensor S is the micro-couple tensor, the vector $\mathbf{z}$ is the vector of the internal body micro-forces and the second order tensor $\mathbf{P}$ is the micro-stress tensor. As shown in Equations (5-8), all these quantities depend on the geometry of the internal phases. In particular, the micro-structural tensor S accounts for the presence of fibres while the two micro-structural measures, $\mathbf{z}$ and $\mathbf{P}$, account for the presence of microcracks. If the microcracks are not considered, the equivalent continuum defined corresponds to a Cosserat continuum.

2.1. Constitutive equations for the multi-field continuum

The response functions assumed for the internal actions of the lattice model are: linear elastic functions for the interacting forces and couples between particles

$$\mathbf{t}_{\mathrm{i}} = \mathbf{K}_{\mathrm{i}}[(\nabla\mathbf{u}-\mathbf{R})(\mathbf{a}-\mathbf{b}) + \nabla\mathbf{R}(\mathbf{a}-\mathbf{x})(\mathbf{p}_{\mathrm{i}}^a-\mathbf{a}) + \nabla\mathbf{R}(\mathbf{b}-\mathbf{x})(\mathbf{p}_{\mathrm{i}}^b-\mathbf{b})] \tag{10}$$

$$\mathbf{C}_{\mathrm{i}} = \mathsf{K}_{\mathrm{i}}^{\mathrm{R}}\nabla\mathbf{R}(\mathbf{a}-\mathbf{b}), \tag{11}$$

where the second order and the third order tensors, $\mathbf{K}_{\mathrm{i}}$ and $\mathsf{K}_{\mathrm{i}}^{\mathrm{R}}$, contain the elastic parameters of the matrix; linear elastic functions for the forces exerted between matrix and microcracks

$$\mathbf{z}_0^h = \mathbf{D}^h(\mathbf{d} + \nabla\mathbf{d}(\mathbf{h}-\mathbf{x})), \tag{12}$$

with the components of the second order tensor $\mathbf{D}^h$ depending on the elastic constants of the matrix and on the microcracks size; non linear functions for the interacting forces between microcracks

$$\mathbf{z}_{\mathrm{j}} = \mathrm{D}\|\mathbf{d} + \nabla\mathbf{d}(\mathbf{h}-\mathbf{x})\|\,\|\mathbf{d} + \nabla\mathbf{d}(\mathbf{k}-\mathbf{x})\|\frac{\mathbf{h}-\mathbf{k}}{\|\mathbf{h}-\mathbf{k}\|^2}, \tag{13}$$

where the constant D depends on the length of the microcracks and on the elastic constants of the matrix. The forces $\mathbf{z}_{\mathrm{j}}$ act in the direction connecting the slits and are analogous to the interacting forces exerted between parallel edge dislocations in elastic media. Finally, others constitutive functions are assumed for the particle-slit interactions

$$\mathbf{r}_{\mathrm{i}} = f(\nabla\mathbf{u}, \mathbf{d}, \nabla\mathbf{d}). \tag{14}$$

Based on the required equivalence in terms of virtual work of the micro and the macro-model, by substituting the Equations (10-14) into the Equations (5-8) the constitutive relationships of the equivalent multi-field continuum are obtained. Accounting for the central material symmetry, that is the basic material symmetry of a periodic, or statistically homogeneous medium, these relations can be expressed as

$$\begin{aligned} \mathbf{S} &= \mathsf{A}(\nabla\mathbf{u}-\mathbf{R}) + \mathsf{D}\nabla\mathbf{d} \\ \mathsf{S} &= \mathsf{F}\nabla\mathbf{R} \\ \mathbf{z} &= \mathsf{H}\mathbf{d} \\ \mathbf{P} &= \mathsf{L}(\nabla\mathbf{u}-\mathbf{R}) + \mathsf{N}\nabla\mathbf{d} + \text{Ø}(\mathbf{d} \bullet \mathbf{d}, \nabla^2\mathbf{d}) \end{aligned} \tag{15}$$

The components of the tensors A, D, F, H, L and N (respectively of the fourth, fourth, sixth, second, fourth and fourth order) and of the function $\boldsymbol{\Psi}$ depend on the elastic constants and on the geometry of the module. In particular, the tensor F contains internal scale parameters of the fibres while the tensor H and the function $\boldsymbol{\Psi}$ include internal scale parameters of the microcracks. These material parameters allow taking properly into accounts size effects.

3. A ONE-DIMENSIONAL SAMPLE

As test sample a one-dimensional problem is studied. A beam, of length L and height H, fixed at one end and subjected to uniformly distributed axial loads, q, along its axis is considered. The material of the beam is characterised by a periodic distribution of fibres, of length $l^{\boldsymbol{a}}$, and of microcracks, of length

l^h. Both fibres and slits are distributed according to the ortho-tetragonal material-symmetry.

Denoting with w, f, and d the relevant components of the macro-displacement, the micro-rotation and the micro-displacement fields, respectively, the balance equations for this problem read

$$
\begin{aligned}
&Aw''=-q/\mathrm{A}\\
&f''=0\\
&Nd''-Hd=0
\end{aligned}
\qquad (16)
$$

where A, H and N are the relevant components of the constitutive tensors **A**, **H** and **N**. For simplicity, the interactions between fibres and microcracks, $\mathbf{r}_i$, and between microcracks, $\mathbf{z}_j$, are not considered. In this case **D** , **L** and **Ψ** are zero.

As boundary conditions the following conditions are assumed

$$
\begin{aligned}
&w(0)=0,\quad w'(0)=qL/(A\mathrm{A})\\
&f(0)=0,\quad f'(0)=0\\
&d(0)=d,\quad d'(0)=0,\qquad \text{with}\ d=pl^h qL^2 r_S/(E\mathrm{A}q),
\end{aligned}
\qquad (17)
$$

where E is the elastic modulus of the matrix, r_S the microcracks density per unit length and q the ratio between the area of the cross section of an ideal elastic string containing the microcracks and the area of the cross section of the truss, A. The overall displacement field of the beam can be easily found as

$$
w+d=\frac{-q}{2A\mathrm{A}}x^2+\frac{qL}{A\mathrm{A}}x+\frac{d}{2}(e^{ax}+e^{-ax}) \qquad (18)
$$

and the micro-rotation field is null.

The values of the elastic parameters in the Equations (16-18) are $A=\mathrm{n}\,E\,\mathrm{H}\,l^a r_F^{\,2}$, with r_F the microfibres density per unit length and n a constant depending on the number and the arrangement of the particles in the module, and $a=\sqrt{H/N}=\mathrm{h}r_S$.

In the case of homogeneous, isotropic, linear-elastic material A=E and δ=0 and the total displacement corresponds to the elastic displacement of the beam, w^e. Fixed a point x of the beam axis, Figure 1a shows the decreasing of the displacement ratio, of the microcracked beam, $(w^e+d)/w^e$, with the microfibres density, r_F, while Figure 1b shows the increasing of the displacement ratio, w/w^e, of the fibre reinforced beam with the increasing of the microcracks density, r_S.

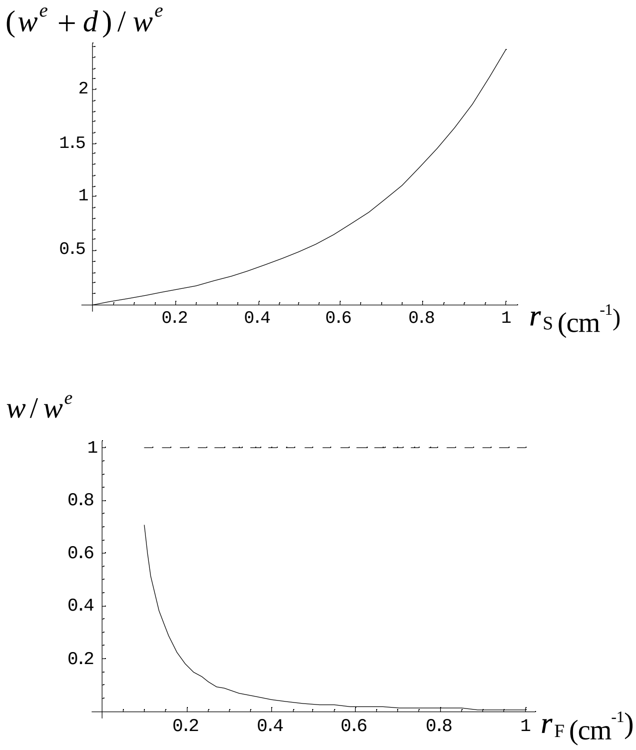

Figure 1. Displacement ratios of the beam versus microcracksdensity (a) and microfibres density (b).

3. References

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