SIZE EFFECTS IN CONTINUUM MODELING OF BRICK MASONRY

R. Masiani[1] and P. Trovalusci[2]

1 ABSTRACT

The linear elastic global behaviour of brick/block masonry is investigated exploiting an 'equivalent' continuum description. The brickwork, regarded as a system of rigid elements interacting by springs, is modeled as a Cauchy and as a Cosserat continuum. The macroscopic elasticities are derived, in terms of microscopic mechanical and geometrical properties, following a procedure of strain energy equivalence. Both the continuos models allow to take into account the bricks shape and arrangement but only the Cosserat continuum accounts for the bricks size. Numerical analyses prove that, for structures which exhibit size effects, the micropolar continuum approach results more effective than the classical one.

2 INTRODUCTION

The mechanical behaviour of masonry made of bricks or stones arranged in various dispositions is influenced by the shape, the size and the geometric texture of the units as much as by the material properties of bricks and mortar. A fine descriptions, based on the discrete modeling of the brick assembly, represents the natural way to take into account these features in the modeling. However, such approach becomes inapplicable for systems with a large number of degrees of freedom, and then the continuum modeling is often required.
The effectiveness of a gross model in simulating the global behaviour of the actual discontinuous material is related to the possibility of including the geometrical and the mechanical properties of the brickwork in the constitutive relationships. In the last twenty years several empirical attempts to establish the average properties of an equivalent continuum are presented. Further analytical formulations, based on homogenization techniques, are provided, for example, in [1, 2, 3, 4, 5]. In such models, masonry is replaced by an equivalent classical continuum whose constitutive laws include the characteristic length measures of the geometrical microstructure only as dimensionless ratios. To satisfy the local periodical boundary conditions, the standard homogenization methods require the dimension of the unit cell, representative of the masonry stack, be much smaller than the relevant dimension of the body. This implies that structures made of similar bricks of different size have

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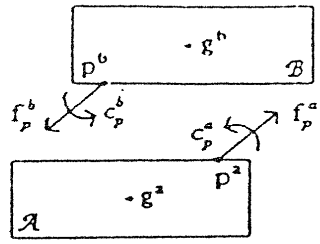

Figure 1: Dynamical interaction in the discrete model

the same behaviour and than the size effects cannot be accounted for. It is quite evident, however, that for problem in which the mutual rotations of the units have a meaningful role – as in presence of concentrated loads, openings or other singularities – the differences between structures made of small or large blocks must be considered. On the other hand, the element of the equivalent continuum, corresponding to particles of finite size, should not have vanishing dimensions and, consequently, contact couples must exist to not violate the moment balance equations (see for instance [6]). Hence, the classical continuum should be used only when the dimension of the bricks are negligible with respect to the scale of the problem.

In a earlier work [7], to take account of the bricks dimensions without violate the internal balance equations, we identified the constitutive functions for the contact actions of a structured Cosserat continuum, using a procedure of strain energy equivalence. This procedure provides the continuum elastic coefficients in terms of contact actions and geometry of the discrete assembly. The micropolar approach, accounting for the bricks relative rotations and the interaction couples, does not require a priori the periodic cell, on which to establish the energy equivalence, be small and revealed itself as one of the available way to obtain continuum constitutive functions depending on a length parameter of the microstructure (see also [8, 9]). To have a comparison, the same equivalence procedure are here used to identify the elasticities of a Cauchy continuum which cannot account for the effect of mutual rotations of the units.

3 MACROSCOPIC DESCRIPTION OF BRICKWORK

The macroscopic constitutive relationships for the Cosserat and the Cauchy continuum models are derived by postulating the equivalence with a discrete model of the brickwork in terms of strain energy density [7]. Some detail of the proposed procedure, limited for simplicity to the two dimensional case, are recalled here below. As first step the attention is focused in the framework of the linear elastic theory.

3.1 Discrete model

In a masonry structure the blocks are generally much less deformable than the joints; then we regard the discrete assembly as a system of rigid bodies connected by two stretch springs, normal and tangential to the contact surface, and one rotational spring. A force, $\mathrm{f}_p = \mathrm{f}_p^a = -\mathrm{f}_p^b$, and a couple, $c_p = c_p^a = -c_p^b$, describe the interactions between two adjacent blocks, $\mathcal{A}$ and $\mathcal{B}$, through two test points assumed coincident in the reference shape: $p^a = p^b = p$ (Fig. 1). Defined the relative displacements of each couple of points, w_p, and the relative rotations between the two blocks, ω_p, as strain measures of the assembly, the linear elastic constitutive relations at contact p take the form

$$\mathrm{f}_p = \mathrm{K}_p \mathrm{w}_p\,, \qquad c_p = k_p^r \omega_p\,, \tag{1}$$

where K_p is the stiffness tensor of the stretch springs and k^r is the twisting stiffness constant. For masonry with periodic fabric, a 'module' $\mathcal{M}$ can be selected so as to be the smallest area element in which all kind of brick interactions are accounted for.The power expended by the contact actions of the module, in any admissible linearised strain field, coincides with the strain energy of the brickwork, whose formula is

$$\Pi(\mathrm{w}^p, \omega^p) = \sum_p \left\{ \mathrm{K}_p \mathrm{w}_p \cdot \mathrm{w}_p + k_p^r \omega_p^2 \right\} \tag{2}$$

3.2 The equivalent Cosserat model

From a kinematical point of view, the micropolar continuum is a system of material points with the translational, u, an the rotational, ϕ, degrees of freedom (microrotation). We assume the following correspondences between the discrete and the continuum displacement fields

$$\mathrm{w}(g^a) = \mathrm{u}(x) + \mathrm{H}(x)(g^a - x)\,, \qquad \omega^a = \phi(x) + \mathrm{h}\cdot(g^a - x)\,, \tag{3}$$

where $\mathrm{w}(g^a)$ and ω^a are respectively the infinitesimal displacement of the center of a generic brick $\mathcal{A}$ and its infinitesimal rotation; $\mathrm{H} = \mathrm{grad}\mathrm{u}$ and $\mathrm{h} = \mathrm{grad}\phi$. Since the discrete strain measures can be derived in terms of $\mathrm{w}(g^a)$ and ω^a, the strain energy (2) of the module can be expressed in terms of the continuum strain measures using (3). These measures are defined as $\mathrm{U} = \mathrm{H} - \mathrm{R}$ and h, where, assumed an orthonormal basis $\{\mathrm{e}_i\}$ in $\mathcal{R}^3$, $\mathrm{R} = \phi\mathrm{Q}$ is the skewsymmetric microrotation tensor and $\mathrm{Q} = (\mathrm{e}_2 \otimes \mathrm{e}_1 - \mathrm{e}_1 \otimes \mathrm{e}_2)$. Note that the relative rotation of the bricks has the microrotation gradient field as continuum counterpart. Then, said $A_{\mathcal{M}}$ the area of the module, by requiring the discrete and the continuum strain energy density be equal

$$\frac{\Pi(\mathrm{U},\mathrm{h})}{A_{\mathcal{M}}} \equiv \mathrm{S}\cdot\mathrm{U} + \mathrm{s}\cdot\mathrm{h} \qquad \forall \mathrm{U},\mathrm{h}, \tag{4}$$

the linear elastic constitutive functions of the stress S and of the couple stress s, can be identified

$$\begin{aligned}
\mathrm{S}(x) = \frac{1}{A_{\mathcal{M}}} \quad \textstyle\sum_p \quad & \{\, \mathrm{K}_p[\mathrm{U}(g^b - g^a) + \mathrm{Q}((g^b - x)\otimes(p - g^b) - (g^a - x)\otimes(p - g^a))\mathrm{h}]\cdot \\
& \otimes (g^b - g^a) \,\} \\
\mathrm{s}(x) = \frac{1}{A_{\mathcal{M}}} \quad \textstyle\sum_p \quad & \{\, [(g^b - x)\otimes \mathrm{Q}(p - g^b) - (g^a - x)\otimes \mathrm{Q}(p - g^a)] \\
& \mathrm{K}_p[\mathrm{U}(g^b - g^a) + \mathrm{Q}((g^b - x)\otimes(p - g^b) - (g^a - x)\otimes(p - g^a))\mathrm{h}] \\
& + k_p^r[(g^b - g^a)\otimes(g^b - g^a)]\mathrm{h} \,\}
\end{aligned} \tag{5}$$

It is important to remark that there are no theoretical obstacles to apply the explained procedure if the discrete constitutive laws are assumed non linear and non conservative.

3.3 Cauchy equivalent continuum

The same equivalence procedure can be exploited to identify the stress-strain relations for a classical equivalent continuum. In this case the rotations of all the blocks of the module are assumed equals and coincident with the local rigid rotation θ of the continuum neighbourhood; that is mutual rotations of units have not place in the continuum model. Then, by postulating the correspondence between the discrete and the continuum displacements as

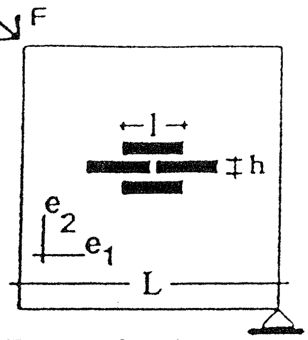

Figure 2: Sample problem

in (3a), by expressing the sole strain measure of the module $\mathbf{w}_p$ as function of the Cauchy strain tensor $\mathbf{E} = \mathrm{sym}(\mathbf{H})$ and by equating the respective strain energy density

$$\frac{1}{A_{\mathcal{M}}}\sum_p \{\mathrm{K}_p \mathbf{w}_p \cdot \mathbf{w}_p\} \equiv \check{\mathrm{S}} \cdot \mathrm{E} \qquad \forall \mathrm{E}, \tag{6}$$

the following constitutive function for the ordinary contact actions holds

$$\check{\mathrm{S}}(x) = \frac{1}{A_{\mathcal{M}}}\sum_p \mathrm{sym}\{\, \mathrm{K}_p[\mathrm{E}(g^b - g^a)] \otimes (g^b - g^a)\,\} \tag{7}$$

4 CONSTITUTIVE EXPRESSIONS AND SIZE EFFECTS

The material properties of the micropolar and classical continuum are identified as functions of the discrete mechanical properties, the components of K and k^r, and of the fabric vectors $(g^b - g^a)$, $(g^a - x)$, $(g^b - x)$, $(p - g^a)$, $(p - g^b)$ which account for the geometry of the module. By ordering the four and the two independent components respectively of the Cosserat stress and couple stress into the vector $\mathbf{t} \in \mathcal{R}^6$, $(\mathbf{t}) = \{S_{11}, S_{22}, S_{12}, S_{21}, s_1, s_2\}^T$, and the independent components of the strain in the vector $\mathrm{d} \in \mathcal{R}^6$, $(\mathrm{d}) = \{U_{11}, U_{22}, U_{12}, U_{21}, h_1, h_2\}^T$, the stress-strain relationship (6) can be represented in the form

$$\mathbf{t} = \mathrm{D}\mathrm{d} \tag{8}$$

Similarly, by ordering the Cauchy stress and strain components in the vectors $\check{\mathbf{t}}, \check{\mathrm{d}} \in \mathcal{R}^3$, with $(\check{\mathbf{t}}) = \{\check{S}_{11}, \check{S}_{22}, \check{S}_{12}\}^T$ and $(\check{\mathrm{d}}) = \{E_{11}, E_{22}, E_{12}\}^T$, the constitutive relation (7) assumes the form

$$\check{\mathbf{t}} = \check{\mathrm{D}}\check{\mathrm{d}} \tag{9}$$

Note that we consider hyperelastic materials for which $\mathrm{K} = \mathrm{K}^T$, so that $\mathrm{D} = \mathrm{D}^T$ and $\check{\mathrm{D}} = \check{\mathrm{D}}^T$. The components of D and $\check{\mathrm{D}}$ as functions of the geometrical and the mechanical parameters of the module are reported in Appendix.
If the brickwork has a periodic structure, the equivalent medium results homogeneous. The continuum material symmetries depend on the material and the geometrical symmetries of the module and they are generally different in the Cosserat and in the Cauchy medium. The symmetric transformations for the discrete correspond to those of the structured continuum while they do not always correspond to those of the classical continuum. For instance, for an orthotropic texture the corresponding equivalent materials are both orthotropic; for a centrosymmetric texture instead, the Cosserat material is also centrosymmetric while the Cauchy material results orthotropic [10].

The influence of the size of the bricks in the two equivalent constitutive models, can be discussed by comparing the expressions of the independent elastic coefficients reported

	Cosserat		Cauchy
D_{11}	$a_1 k_n \frac{l}{h} + a_2 k_t (\frac{l}{h})^2$	$\breve{D}_{11}$	$a_1 k_n \frac{l}{h} + a_2 k_t (\frac{l}{h})^2$
D_{22}	$a_1 k_n$	$\breve{D}_{22}$	$a_1 k_n$
D_{12}	0	$\breve{D}_{12}$	0
D_{33}	$a_1 k_t$	$\breve{D}_{33}$	$\frac{1}{2}(a_2 k_n (\frac{l}{h})^2 + a_1 (1 + \frac{l}{h}) k_t)$
D_{44}	$a_2 k_n (\frac{l}{h})^2 + a_1 k_t \frac{l}{h}$		
D_{34}	0		
D_{55}	$a_3 k_n \frac{l^4}{h^2} + a_4 k_n l h + a_5 k_t l^2$		
D_{66}	$a_6 k_n l^2 + a_7 k_t h^2$		
D_{56}	0		

Table 1: Constitutive terms for the module of Fig. 2. l and h are the dimensions of the brick.

in Tab. (1), obtained for the orthotropic texture of Fig. 2. In the table k_n and k_t are respectively the normal and the tangential stiffness per unit length and unit thickness of the joints and a_i, are constants depending on the bricks disposition. The diagonal structure of $\mathbf{D}$ and $\breve{\mathbf{D}}$ is due to the particular material and geometrical symmetries of the module. For a masonry with periodical texture at least centrosymmetric, the terms which relate the ordinary stress to the microrotation gradient (D_{5j} and $D_{6j}, j = 1,4$) are zero. Moreover, we assume that the joints do not exhibit dilatancy, so that there is not Poisson effect and $D_{12} = \breve{D}_{12} = 0$.
The distinct feature of the micropolar model is that the twisting stiffness terms, elasticities which relates the components of the microrotation gradient to the couple stress, are functions of the bricks dimensions and not only of their ratio. This implies that for problems in which the influence of the brick size are relevant the structured continuum only, tacking into account the mutual rotations of the units, can gives satisfactory results. Note that, when the size of the units vanishes, the couple stress vanishes whichever is the microrotation gradient. Consequently, from balance, the stress tensor S becomes symmetric but, due to the different tangential moduli, D_{12} and D_{21}, the strain tensor U remains unsymmetric. This means that the Cosserat continuum does not behave as a Cauchy continuum even if the relevant dimension of the microstructure vanishes. Also the numerical results discussed in the next section confirm that the micropolar and the discrete solutions do not converge to the Cauchy solution when the brick size decreases. It can be shown that this convergence occurs only for orthotetragonal materials [10]. Hence, the Cauchy continuum appears suitable only for problems in which the size of the brick are not relevant and for media with very particular material symmetries.

5 FINITE ELEMENT FORMULATION

The continuum micropolar problem is solved numerically using a F. E. discretisation in three nodes triangular plane elements with three degrees of freedom per node: two translations and one in-plane rotation (drilling freedom). Here below some detail of the implemented element are given.
Let the orthonormal base $\{\mathbf{e}_i\}$ in $\mathcal{R}^3$ be the element local frame and $\{\mathbf{e}'_i\}$ an orthonormal base in $\mathcal{R}^9$. We denote $\mathbf{v}_n = \sum_{\alpha=1}^{2} u_{n\alpha}\mathbf{e}_\alpha + \phi_n \mathbf{e}_3$ $(n = 1,3)$ as the displacement vector of the nth-node and

$$\mathbf{v} := \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{e}'_{3(i-1)+j} \otimes \mathbf{e}_j \mathbf{v}_i \tag{10}$$

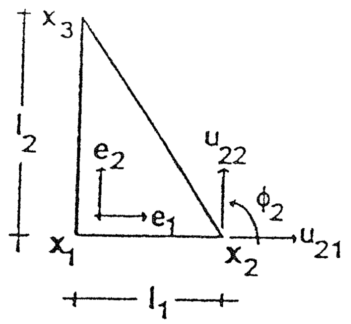

Figure 3: Triangular finite element

as the element displacement vector in $\mathcal{R}^9$. Assuming that the displacement field $q(x) = \sum_{\alpha=1}^{2} u_\alpha(x)e_\alpha + \phi(x)e_3$ is interpolated by a linear shape function, the compatibility condition $q(x_\kappa) = v_\kappa$ leads to

$$q = \Phi(x)v, \qquad \text{with } \Phi(x) := \sum_{i=1}^{3} A_i \sum_{j=1}^{3} e_j \otimes e'_{3(i-1)+j} \tag{11}$$

where, said l_1 and l_2 the the length of the element sides if Fig. 3, $A_1 = 1 - x_1/l_1 - x_2/l_2$; $A_2 = x_1/l_1$; $A_3 = x_2/l_2$. The plane strain vector d is then derived in terms of nodal displacements as

$$d = \nabla\Phi v = Bv \tag{12}$$

where the matrix associated to ∇ is

$$(\nabla) = \begin{pmatrix} \partial_1 & 0 & \partial_2 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & 1 & -1 & \partial_1 & \partial_2 \end{pmatrix}^T$$

It can be noted that the two angular distortions U_{12} and U_{21} are linear functions of the element coordinates while the other strains components are constant. For the solution, we used the standard equilibrium approach which requires the variation of the potential energy

$$\delta\mathcal{E} = \int_B \delta v \cdot B^T DB dA + \int_{\partial B} \delta v \cdot \Phi f ds \tag{13}$$

be zero for any δv, where f is the vector of forces and couples acting along the element boundary. The derived element is a conforming element with displacements functions automatically continuous along the boundary which also exhibit zero nodal actions in any rigid-body motion. The components of its 9 x 9 element stiffness matrix (K_e) as functions of the geometrical and mechanical parameters of the module, where as usage

$$K_e = \int_B B^T DB dA$$

6 TEST PROBLEM

We studied the sample problem sketched in Fig. 2. Three different solutions are compared in Fig. 4. On the left column are plotted the results of the discrete approach, obtained by means of a standard general purpose F. E. program, using rigid constraint equations for the bricks and two nodes elastic springs for the joints. The results from the Cosserat continuum

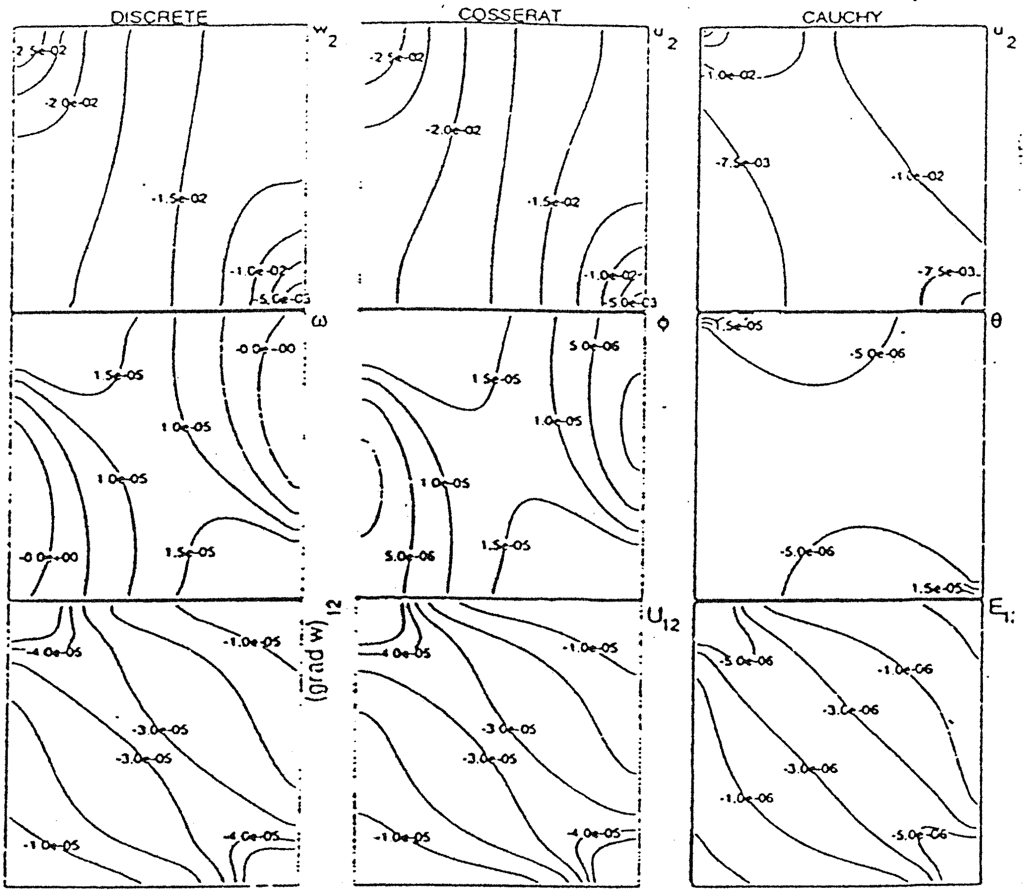

Figure 4: Contour lines of the results from the three models.

are in the second column, evaluated by means of the F.E. discretisation described in the previous section. The solution of the classical continuum model, in the right column, are computed using a standard F.E. program. We drown the contour lines of three significative results: the vertical component of the displacement, w_2, the rotation of the bricks, ω (compared to the microrotation of the Cosserat solution, ϕ, and to the infinitesimal rigid rotation of the Cauchy continuum, θ) and the shearing strain on the direction e_1 and e_2. The Cosserat solution well agrees with the discrete results. On the contrary, the classical continuum is unable to account for the two different shear stiffness, in the directions parallel and normal to the bed joints.
The effect of the bricks size is clearly proved in the diagram of Fig. 5 which reports, for the previous problem, the (normalized) strain energy for different values of the brick length l. The results strictly depend on the brick size in the discrete and micropolar solution, while the Cauchy solution, as stated in the previous section, is constant. Note that this solution has not a clear physical meaning; it not even corresponds to the case of very small blocks.

7 CONCLUSION

The suitability of the continuum modeling of brick masonry is evident whenever a fine description becomes too heavy, but the gross model must hold memory of all the properties

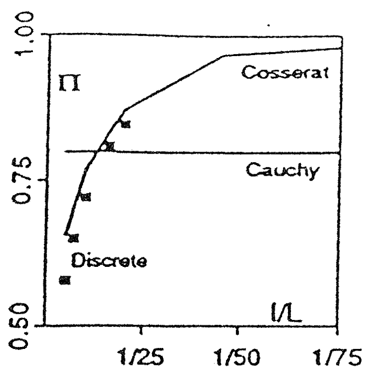

Figure 5: Size effects

relevant to the global behaviour. The proposed equivalence procedure provides a simple way to derive the continuum parameters directly from the mechanical and the geometrical properties of the wall. However, only a micropolar equivalent continuum always exhibits the same material symmetry group of the discrete assemblage and only a Cosserat continuum, not a classical one, is able to take into account the dimension of the bricks through the couple stress. In many cases, depending on the symmetry group of the discrete, a Cauchy equivalent model gives unsatisfactory results. This occurs also if the dimension of the brick is negligible with respect to a dimension of the panel because, anyway, the classical material is unable to differentiate the shear stiffness of the bed joints from those of the head joints. The numerical analyses, performed by means of a E.F. discretization, confirm these statements. Obviously, a realistic description of the masonry behaviour requires more complex constitutive functions, namely non-linear and non-elastic, but they must in any cases take into account the size effects.

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APPENDIX

The representation in terms of components of the constitutive functions in the two-dimensional Cosserat and Cauchy frame are respectively

$$\mathrm{t}_i = \mathrm{D}_{ij}\mathrm{d}_j, \qquad (i,j=1,6) \qquad \text{and} \qquad \tilde{\mathrm{t}}_i = \tilde{\mathrm{D}}_{ij}\tilde{\mathrm{d}}_j, \qquad (i,j=1,3)$$

Since we assume that the tensor K is symmetric we get hyperelastic materials for which the major symmetries of D and $\tilde{\mathrm{D}}$ hold. Let $(g^b-g^a)=\mathrm{g}$, $(g^a-x)=\mathrm{x}$, $(g^b-x)=\mathrm{y}$, $(p-g^a)=\mathrm{a}$, $(p-g^b)=\mathrm{b}$ and $A_{\mathcal{M}}=A$, the explicit expressions for the independent components of the Cosserat elasticity tensors result

$$D_{11}=\frac{1}{A}\sum_p K_{p11}g_1^2 \qquad D_{12}=\tfrac{1}{A}\textstyle\sum_p K_{p12}g_1g_2 \qquad D_{13}=\frac{1}{A}\sum_p K_{p11}g_1g_2$$

$$D_{14}=\frac{1}{A}\sum_p K_{p12}g_1^2 \qquad D_{22}=\tfrac{1}{A}\textstyle\sum_p K_{p22}g_2^2 \qquad D_{23}=\frac{1}{A}\sum_p K_{p21}g_2^2$$

$$D_{24}=\frac{1}{A}\sum_p K_{p22}g_1g_2 \qquad D_{33}=\tfrac{1}{A}\textstyle\sum_p K_{p11}g_2^2 \qquad D_{34}=\frac{1}{A}\sum_p K_{p12}g_1g_2$$

$$D_{44}=\tfrac{1}{A}\textstyle\sum_p K_{p22}g_1^2$$

$$D_{15}=\frac{1}{A}\sum_p g_1\{\,K_{p11}(-y_1b_2+x_1a_2)+K_{p21}(y_1b_1-x_1a_1)\,\}$$

$$D_{16}=\frac{1}{A}\sum_p g_1\{\,K_{p11}(-y_2b_2+x_2a_2)+K_{p12}(y_2b_1-x_2a_1)\,\}$$

$$D_{25}=\frac{1}{A}\sum_p g_2\{\,K_{p21}(-y_1b_2-x_1a_2)+K_{p22}(y_1b_1-x_1a_1)\,\}$$

$$D_{26}=\frac{1}{A}\sum_p g_2\{\,K_{p21}(-y_2b_2+x_2a_2)+K_{p22}(y_2b_1-x_2a_1)\,\}$$

$$D_{35}=\frac{1}{A}\sum_p g_2\{\,K_{p11}(-y_1b_2+x_1a_2)+K_{p12}(y_1p_1+x_1p_1)\,\}$$

$$D_{36}=\frac{1}{A}\sum_p g_2\{\,K_{p11}(-y_2b_2+x_2a_2)+K_{p12}(y_2b_1+x_2a_1)\,\}$$

$$D_{45}=\frac{1}{A}\sum_p g_1\{\,K_{p21}(-y_1b_2+x_1a_2)+K_{p22}(y_1b_1-x_1a_1)\,\}$$

$$D_{46}=\frac{1}{A}\sum_p g_1\{\,K_{p21}(-y_2b_2+x_2a_2)+K_{p22}(y_2b_1-x_2a_1)\,\}$$

$$D_{55} = \frac{1}{A}\sum_p \{ \; K_{p11}(y_1^2 b_2^2 + x_1^2 a_2^2 - 2y_1 x_1 b_2 a_2)$$
$$+ \; K_{p12}(-y_1^2 b_1 b_2 - x_1^2 a_1 a_2 + y_1 x_1 b_1 a_2 + y_1 x_1 b_2 a_1)$$
$$+ \; K_{p21}(-y_1^2 b_1 b_2 - x_1^2 a_1 a_2 + y_1 x_1 b_2 a_1 + y_1 x_1 b_1 a_2)$$
$$+ \; K_{p22}(y_1^2 b_1^2 + x_1^2 a_1^2 - 2y_1 x_1 b_1 a_1) + k_p^r g_1^2\}$$

$$D_{56} = \frac{1}{A}\sum_p \{ \; K_{p11}(y_1 y_2 b_2^2 + x_1 x_2 a_2^2 - y_2 x_1 ba - y_1 x_2 ba)$$
$$+ \; K_{p12}(-y_1 y_2 b_1 b_2 - x_1 x_2 a_1 a_2 + y_2 x_1 b_1 a_2 + y_1 x_2 b_2 a_1)$$
$$+ \; K_{p21}(-y_1 y_2 b_1 b_2 - x_1 x_2 a_1 a_2 + y_2 x_1 b_2 a_1 + y_1 x_2 b_1 a_2)$$
$$+ \; K_{p22}(y_1 y_2 b_1^2 + x_1 x_2 a_1^2 - y_2 x_1 b_1 a_1 - y_1 x_2 b_1 a_1) + k_p^r g_1 g_2\}$$

$$D_{66} = \frac{1}{A}\sum_p \{ \; K_{p11}(y_2^2 b_2^2 + x_2^2 a_2^2 - 2y_2 x_2 ba)$$
$$+ \; K_{p12}(-y_2^2 b_1 b_2 - x_2^2 a_1 a_2 + y_1 x_1 b_1 a_2 + y_1 x_1 b_2 a_1)$$
$$+ \; K_{p21}(-y_1^2 b_2 b_1 - x_2^2 a_1 a_2 + y_2 x_2 b_2 a_1 + y_2 x_2 b_1 a_2)$$
$$+ \; K_{p22}(y_2^2 b_1^2 + x_2^2 a_1^2 - 2y_2 x_2 p_1 p_1) + k_p^r g_2^2\}$$

For the Cauchy elasticity tensor we we obtain $\check{D}_{11} = D_{11}$; $\check{D}_{12} = D_{12}$; $\check{D}_{13} = D_{13} + D_{14}$; $\check{D}_{22} = D_{22}$; $\check{D}_{23} = D_{23} + D_{24}$; $\check{D}_{11} = \frac{1}{2}(D_{33} + 2D_{34} + D_{44})$.